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Soliton Solutions for Time Fractional Hamiltonian System

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, soliton solutions of a fractional partial differential equations using modified extended tanh method with Riccati equation have been proposed. This method is applied to obtain solitary wave solution for the nonlinear time fractional Hamiltonian system. The system is converted into a system of ordinary differential equations using fractional complex transforms and the properties of modified Riemann-Liouville derivative. The proposed technique is concise and easily applicable for solving wide types of time-fractional partial differential equations.

Keywords: Fractional Hamiltonian system; modified extended tanh method.

1 Introduction

As it is well-known that the vast majority of real world problems can only be solved numerically, numerical techniques are used extensively. Fractional differential equations (or extraordinary differential equations), which are a generalization of classical integer order ordinary differential equations, used in the modeling of many problems from different areas of study such as physics, mechanics, plasma physics, dynamical system, signal processing, electricity, finance, biology, and control theory [1-4], necessitated also developing

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methods that adapted them to solve these problems, numerically. Because of their applications, several methods have been introduced to obtain the exact and numerical solutions for the ordinary and partial differential equations [5 -13].

Although fractional calculus is dating back to 17th century, when it was mentioned, for the first time, by Leibniz in a letter to l'Hospital in 1695 [14], it has flourished during the last decades. Recently, numbers of studies have been introduced regarding the fractional phenomena which is related with the applied fields, for instance, exp-function method [15], ansatz method [16-18], first integral method [19-24], functional variable method [25-28], Kudryashov method [29-30], and homotopy perturbation [31-34].

Amongst the fields of research of considerable importance concerning plasma physics, semiconductors and fluid dynamics, is theory of solutions, which serves as a bridge to physics, mathematical engineering, and computer science. In particular, studies of solitary waves theory have attracted intensive interest from mathematicians and physicists. Recently, the area of space time fractional differential equations has been studied by a number of researchers such as, K. Hosseini et al. [35-36], M. Kaplan et al. [37], and M. Eslami [38]. The rest of this paper is constructed as follows: In Sections 2, some basic definitions and some properties of the fractional calculus theory is presented. In Section 3, analysis of the method is given to demonstrate how fractional differential equations are converted into integer-order differential equations. In Section 4, the proposed modified extended tanh method is applied to obtain the exact solutions for the time fractional Hamiltonian system. Sections 5 conclude the paper.

2 Basic Definitions

The Jumarie's modified Riemann-Liouville derivative of order α [39] is defined as:

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-t)^{-\alpha} \left(f(t) - f(0)\right) dt, & 0 < \alpha < 1, \\ \left(f^{(n)}(x)\right)^{(\alpha-n)}, \ n \le \alpha < n+1, \ n \ge 1. & \end{cases} \tag{1}$$

where $\Gamma(x)$ is the Gamma function which is defined as

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$$

Some properties of the Jumarie's modified Riemann-Liouville derivative are listed below:

$$D_x^{\alpha} x^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} x^{r-\alpha} \tag{2}$$

$$D_x^{\alpha} \left(a f(x) + b g(x)\right) = a D_x^{\alpha} f(x) + b D_x^{\alpha} g(x) \tag{3}$$

where a and b are constants.

$$D_x^{\alpha} f(\xi) = \frac{df}{d\xi} D_x^{\alpha}(\xi) \tag{4}$$

where $\xi = g(x)$ [40].

3 Analysis of the Method

In order to describe the principle idea of this method, consider the following nonlinear fractional differential equation:

$$F\left(u, D_t^{\alpha_1}u, D_x^{\alpha_2}u, D_{tt}^{2\alpha_1}u, D_{xx}^{2\alpha_2}u, D_t^{\alpha_1}D_x^{\alpha_2}u, \ldots\right), \qquad 0<\alpha_1, \alpha_2<1 \tag{5}$$

the fractional complex transformation

$$u(x,t)=f(\xi) \quad , \quad \xi=\frac{k}{\Gamma(1+\alpha_2)}x^{\alpha_2}-\frac{c}{\Gamma(1+\alpha_1)}t^{\alpha_1}-x_0$$

where k, c are non-zero constants and x_0 is arbitrary constant, converts equation (5) into an integer order nonlinear ordinary differential equation:

$$H(f, f', f'', f''', \ldots)=0 \tag{6}$$

where the derivatives are with respect to ξ. Now, the solution of (6) is presented as a finite series:

$$f(\xi)=a_0+\sum_{n=1}^{N}\left(a_n\varphi^n(\xi)+b_n\varphi^{-n}(\xi)\right) \tag{7}$$

where a_n, b_n, $n=1,2,\ldots\ldots,N$ are constants which will be evaluated, and $\varphi(\xi)$ satisfies the following Riccati equation:

$$\varphi'=b+\varphi^2 \tag{8}$$

where b is constant. Equation (8) has the following general solutions:

(i) If $b<0$, then
$\varphi=-\sqrt{-b}\tanh\left(\sqrt{-b}\xi\right)$, or $\varphi=-\sqrt{-b}\coth\left(\sqrt{-b}\xi\right)$
(ii) If $b>0$, then
$\varphi=\sqrt{b}\tan\left(\sqrt{b}\xi\right)$, or $\varphi=-\sqrt{b}\cot\left(\sqrt{b}\xi\right)$
(iii) If $b=0$, then
$\varphi=\frac{-1}{\xi}$.

The parameter N is usually determined by balancing the linear and nonlinear terms of highest orders in (5). Substituting Eq. (7) and their necessary derivatives, for example:

$$f'=\sum_{n=1}^{N}\left(a_n n\varphi^{n-1}\left(b+\varphi^2\right)-b_n n\varphi^{-n-1}\left(b+\varphi^2\right)\right)$$

$$f''=\sum_{n=1}^{N}\begin{pmatrix}a_n n(n-1)\varphi^{n-2}\left(b+\varphi^2\right)^2+2na_n\ \varphi^n\left(b+\varphi^2\right)+b_n n(n+1)\varphi^{-n-2}\left(b+\varphi^2\right)^2- \\ 2b_n n\varphi^{-n}\left(b+\varphi^2\right)\end{pmatrix}$$

into (5) gives

$$P(\varphi(\xi))=0 \tag{9}$$

where $P(\varphi(\xi))$ is polynomial in $\varphi(\xi)$. By equating the coefficient of each power of $\varphi(\xi)$ in (9) to zero, a system of algebraic equations will be obtained which yields the exact solution of (5).

4 Applications

Consider the Hamiltonian system with time fractional derivatives:

$$\begin{aligned} D_t^\alpha u(x,t) &= D_x u(x,t)+2v(x,t) \\ D_t^\alpha v(x,t) &= 2\mu u(x,t)v(x,t) \end{aligned} \tag{10}$$

where $0<\alpha\le 1$, and μ is a real parameter.

To obtain a solution $u(x,t), v(x,t)$ of system (10), the following fractional complex transformation is used:

$$u(x,t)=f(\xi), \qquad v(x,t)=g(\xi) \qquad \xi = kx-\frac{c}{\Gamma(1+\alpha)}t^\alpha - x_0 \tag{11}$$

where 'Γ'' is Gamma function, k, c , and x_0 are constants.

Using equation (11), we get:

$$D_t^\alpha u=-c\,f'(\xi), \qquad D_t^\alpha v=-c g'(\xi), \qquad D_x u = k\,f'(\xi), \quad ,$$

Then the equations in (10) are reduced into integer order nonlinear ordinary differential equations as presented below:

$$-cf'=kf'+2g \tag{12}$$

$$-cg'=2\mu fg \tag{13}$$

From eq. (12) we get:

$$g=-\frac{1}{2}(c+k)f' \tag{14}$$

Substituting equation (14) into equation (13), gives:

$$cf''+2\mu ff'=0 \tag{15}$$

Integrating (15) once with respect to ξ , we get:

$$cf'+\mu f^2=A \tag{16}$$

Where A is the constant of integration.

4.1 Exact solutions of the Hamiltonian system

Balancing the nonlinear term of highest order with the highest order linear term in Equation. (12), leads to $N = 1$. This offers the following series:

$$f(\xi) = a_0 + a_1 \varphi(\xi) + b_1 \varphi^{-1}(\xi) \tag{17}$$

By substituting (17) in (16), the following system of algebraic equations is obtained:

$$ca_1 b - cb_1 + \mu a_0^2 + 2\mu a_1 b_1 = A$$
$$2\mu a_0 a_1 = 0$$
$$ca_1 + \mu a_1^2 = 0$$
$$2\mu a_0 b_1 = 0$$
$$-cbb_1 + \mu b_1^2 == 0$$

By solving the above system, the following solution is resulted.

Case 1:

$$a_0 = a_1 = 0,\ b = \frac{-\mu A}{c^2},\ b_1 = -\frac{A}{c}$$

Hence:

$$u_1(x,t) = -\frac{A}{\sqrt{b}\, c}\cot(\sqrt{b}\,\xi),\ b > 0$$
$$v_1(x,t) = -\frac{A(c+k)}{2c}\csc^2(\sqrt{b}\,\xi),\ b > 0$$
$$u_2(x,t) = \frac{A}{\sqrt{b}\, c}\tan(\sqrt{b}\,\xi),\ b > 0$$
$$v_2(x,t) = -\frac{A(c+k)}{2c}\sec^2(\sqrt{b}\,\xi),\ b > 0$$
$$u_3(x,t) = \frac{A}{\sqrt{-b}\, c}\coth(\sqrt{-b}\,\xi),\ b < 0$$
$$v_3(x,t) = \frac{-A(c+k)}{2c}\csc h^2(\sqrt{-b}\,\xi),\ b < 0$$
$$u_4(x,t) = \frac{A}{\sqrt{-b}\, c}\tanh(\sqrt{-b}\,\xi),\ b < 0$$
$$v_4(x,t) = -\frac{A(c+k)}{2c}\sec h^2(\sqrt{-b}\,\xi),\ b < 0$$

Where

$$\xi = kx - \frac{c}{\Gamma(1+\alpha)}t^{\alpha} - x_0, \qquad b = \frac{-\mu A}{c^2}$$

Case 2:

$$a_0 = b_1 = 0,\ b = \frac{-\mu A}{c^2},\ a_1 = -\frac{c}{\mu}$$

Hence:

$$u_5(x,t) = -\frac{c}{\mu}\sqrt{b}\tan(\sqrt{b}\,\xi),\ b > 0$$

$$v_5(x,t) = \frac{cb(c+k)}{2\mu}\sec^2(\sqrt{b}\,\xi),\ b > 0$$

$$u_6(x,t) = \frac{c}{\mu}\sqrt{b}\cot(\sqrt{b}\,\xi),\ b > 0$$

$$v_6(x,t) = \frac{cb(c+k)}{2\mu}\csc^2(\sqrt{b}\,\xi),\ b > 0$$

$$u_7(x,t) = \frac{c}{\mu}\sqrt{-b}\tanh(\sqrt{-b}\,\xi),\ b < 0$$

$$v_7(x,t) = \frac{cb(c+k)}{2\mu}\sec h^2(\sqrt{-b}\,\xi),\ b < 0$$

$$u_8(x,t) = \frac{c}{\mu}\sqrt{-b}\coth(\sqrt{-b}\,\xi),\ b < 0$$

$$v_8(x,t) = -\frac{cb(c+k)}{2\mu}\csc h^2(\sqrt{-b}\,\xi),\ b < 0$$

Where

$$\xi = kx - \frac{c}{\Gamma(1+\alpha)}t^{\alpha} - x_0,\ b = -\frac{\mu A}{c^2}$$

Case 3:

$$a_0 = 0,\ a_1 = -\frac{c}{\mu},\ b_1 = -\frac{A}{4c},b = -\frac{\mu A}{4c^2},$$

Hence:

$$u_9(x,t) = \frac{c\sqrt{-b}}{\mu}\tanh(\sqrt{-b}\xi) + \frac{A}{4c\sqrt{-b}}\coth(\sqrt{-b}\xi),\ b < 0$$

$$v_9(x,t) = \frac{cb(c+k)}{2\mu}\sec h^2(\sqrt{-b}\xi) + \frac{A(c+k)}{8c}\csc h^2(\sqrt{-b}\xi),\ b < 0$$

$$u_{10}(x,t) = \frac{c\sqrt{-b}}{\mu}\coth(\sqrt{-b}\xi) + \frac{A}{4c\sqrt{-b}}\tanh(\sqrt{-b}\xi),\ b < 0$$

$$v_{10}(x,t) = -\frac{cb(c+k)}{2\mu}\csc h^2(\sqrt{-b}\xi) - \frac{A(c+k)}{8c}\sec h^2(\sqrt{-b}\xi),\ b < 0$$

$$u_{11}(x,t)=-\frac{c\sqrt{b}}{\mu}\tan(\sqrt{b}\xi)-\frac{A}{4c\sqrt{b}}\cot(\sqrt{b}\xi),\, b>0$$

$$v_{11}(x,t)=\frac{cb(c+k)}{2\mu}\sec^2(\sqrt{b}\xi)-\frac{A(c+k)}{8c}\csc^2(\sqrt{b}\xi),\, b>0$$

$$u_{12}(x,t)=\frac{c\sqrt{b}}{\mu}\cot(\sqrt{b}\xi)+\frac{A}{4c\sqrt{b}}\tan(\sqrt{b}\xi),\;\; b>0,$$

$$v_{12}(x,t)=\frac{cb(c+k)}{2\mu}\csc^2(\sqrt{b}\xi)-\frac{A(c+k)}{8c}\sec^2(\sqrt{b}\xi),\;\; b>0,$$

Where

$$\xi = kx-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}-x_0,\, b=\frac{c}{k^3}$$

Now, we plot these solutions at different time levels and different values of α , and we can show the motion of solitary waves in Figs. 1, 2, 3 and 4.

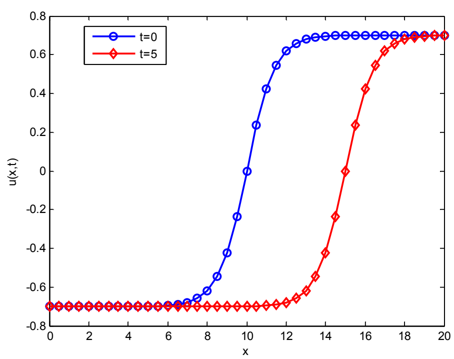

Fig. 1(a). α =1

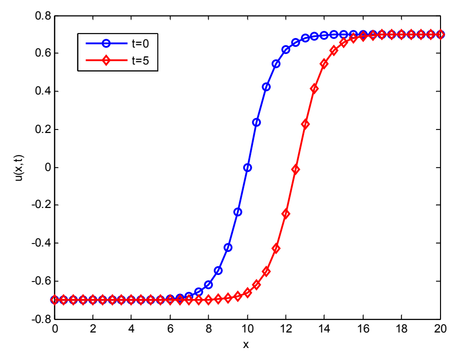

Fig. 1(b). α =0.5

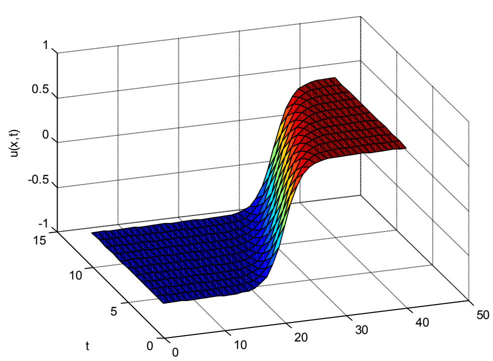

Fig. 1(c). α =1

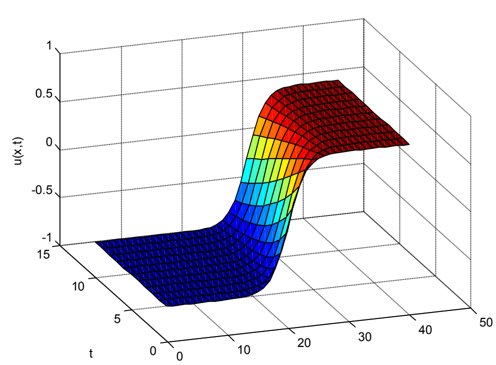

Fig. 1(d). α =0.5

Fig. 1. Plots of $u(x,t)$ **with** $k=c=\mu=1,\, A=0.5\;\; x_0=10, 0\le x\le 20$**, at different time levels and** $\alpha=0.5,1$

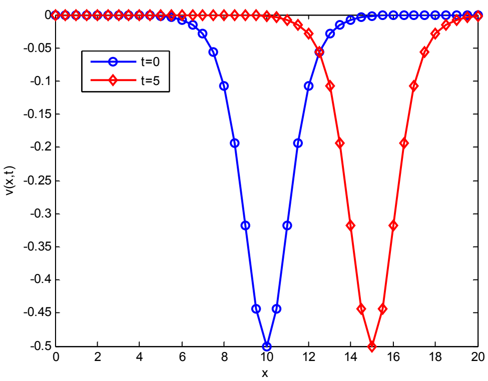

Fig. 2(a). $\alpha = 1$

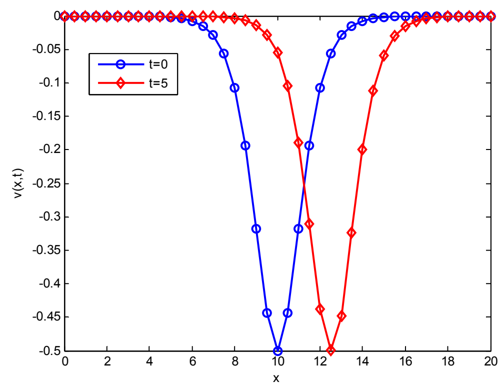

Fig. 2(b). $\alpha = 0.5$

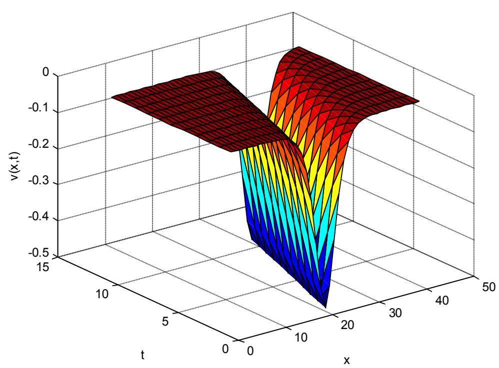

Fig. 2(c). $\alpha = 1$

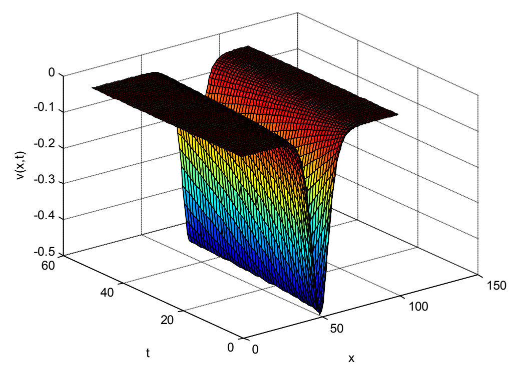

Fig. 2(d). $\alpha = 0.5$

Fig. 2. Plots of $v(x,t)$ **with** $k = c = \mu = 1,\ A = 0.5\ \ x_0 = 10, 0 \le x \le 20$**, at different time levels and** $\alpha = 0.5, 1$

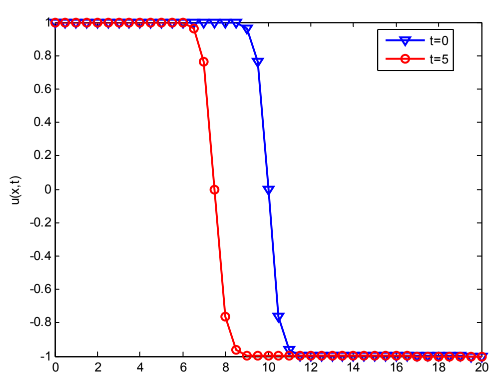

Fig. 3(a). $\alpha = 1$

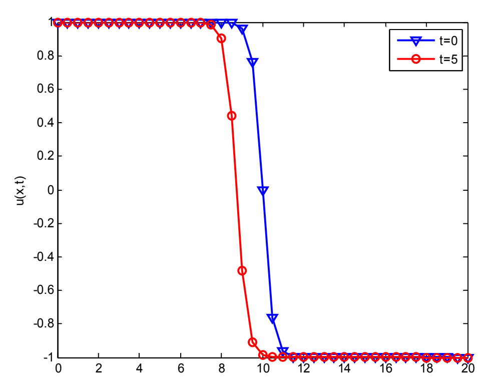

Fig. 3(b). $\alpha = 0.5$

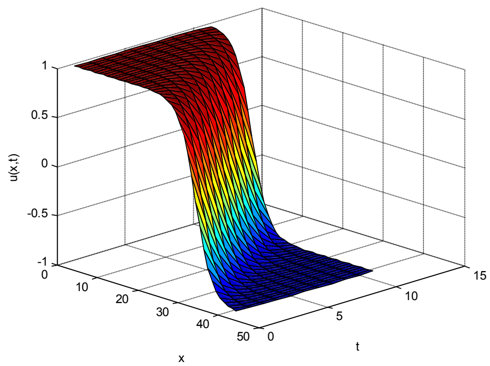

Fig. 3(c). $\alpha = 1$

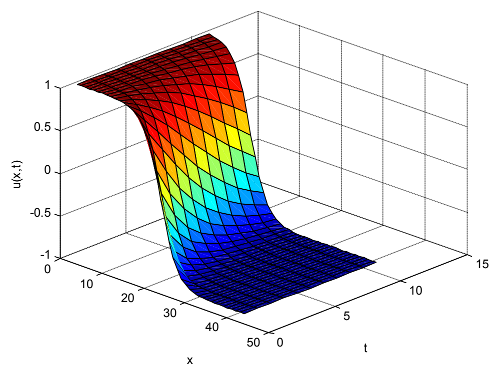

Fig. 3(d). $\alpha = 0.5$

Fig. 3. Plots of $u(x,t)$ **with** $k = A = \mu = 1, c = -0.5 \quad x_0 = 5, 0 \le x \le 10$**, at different time levels and** $\alpha = 0.5, 1$

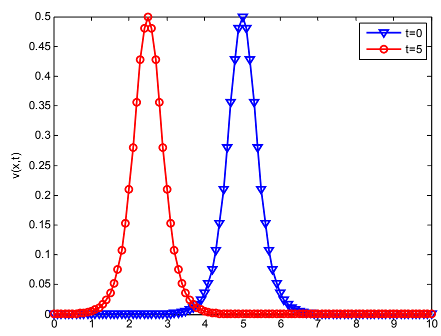

Fig. 4(a). $\alpha = 1$

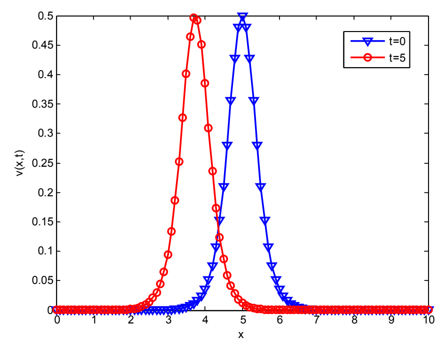

Fig. 4(b). $\alpha = 0.5$

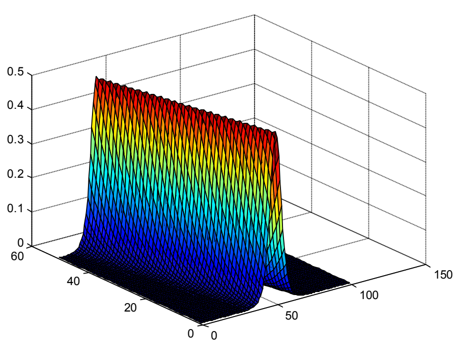

Fig. 4(c). $\alpha = 1$

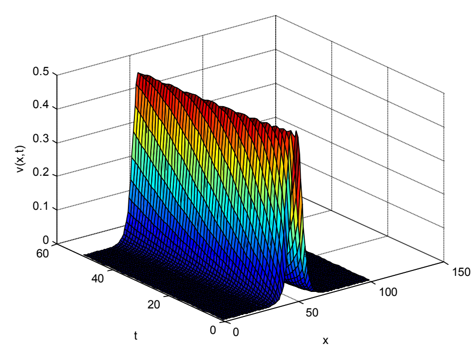

Fig. 4(d). $\alpha = 0.5$

Fig. 4. Plots of $v(x,t)$ **with** $k = A = \mu = 1, c = -0.5 \quad x_0 = 5, 0 \le x \le 10$**, at different time levels and** $\alpha = 0.5, 1$

5 Conclusion

In this research, Soliton solutions of the time fractional Hamiltonian system have been obtained using modified extended tanh method with Riccati equation. The results showed that the method is a powerful and an efficient method. This method is simple and concise. We conclude that the proposed method can be used to solve other linear and nonlinear fractional partial differential equations in engineering and mathematical physics.

Competing Interests

Author has declared that no competing interests exist.

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