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Subharmonic Solutions of Governed MEMS System Subjected to Parametric and External Excitations

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

Subharmonic periodic solutions of order $(\frac{1}{2}, \frac{1}{4})$ to a weakly second order ordinary differential equation which governed the motion of a micro-dynamical system are studied analytically. Applying the method of multiple scales, we derive the modulation equation in the amplitude and the phase of each type of periodic solutions. Determine the steady-state solutions (fixed-points of the modulation equation). Obtained the frequency-response equation (The relation between the amplitude and the detuning parameter and other parameter in the differential equation). Stability analysis of the steady-state solutions is given. Numerical study of the frequency-response equation are carried out. The results are presented in a group of Figures in which solid (dashed) curves indicated stable (unstable) preiodic solutions.

Keywords: MEMS; Multiple scales method; parametric excitation and external excitation.

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1 Introduction

Micro-electro-mechanical system (MEMS) actuated electrostatically, find many applications in different fields, for example, switches, inertial sensors, and relays. The instability of the motion of this devices which results from the interaction between the elastic and electrostatic forces is the aim of many works [1, 2, 3].

Fauzi et al. [4] studied the suppression of pull instability in MEMS. By taking the elastic restoring force as a linear function of the displacement and applying the method of direct partition of the motion [19] to slow and fast motion, and then periodic solution of the slow-motion, represent the solution of the original equations. In this paper, we take the elastic restoring force as a weakly nonlinear function in the displacement contains quadratic and cubic terms. The mathematical model of the oscillatory motion of MEMS is a weakly nonlinear second order differential equation with a weakly nonlinear parametric term, and external excitation term, which is known by a modified Duffing equation, subjected to parametric and external excitation. The periodic solutions of type subharmonic of even order ($\frac{1}{2}, \frac{1}{4}$) are the object of this paper. The periodic solutions of the governed equation (The response of the corresponding dynamical system) are the object of many investigators [5, 6].

From the literature, it is known that a Duffing equation models the dynamical behavior of many dynamical system. Applying the perturbation method (Multiple Scales Method)(MMS) [7, 8]. Belhaq [9] studied numerical study for parametric excitation of differential equation near a 4-resonance. A nonlinear parametric feedback control is suggested to modify the steady-state solutions responses thus to reduce the amplitude of the response and to remove the saddle-node bifurcation [10], and an odd nonlinearity problem is treated using MMS I and MMS II modified [11]. Maccari considered the bifurcation control for the forced Zakharov-Kusnetsov equation using delay feedback linear control terms [12]. The linear feedback time is designed to modify the associated Jacobian matrix of the system, thus delaying the occurrence of unwanted bifurcations but the nonlinear term is used to suppress subcritical and supercritical bifurcations, hence stabilizing the bifurcations [13]. Elnaggar et al. studied superharmonic, harmonic and subharmonic for weakly nonlinear second order differential equation [14, 15]. From literature, it is known that a Duffing model may describe nonlinear effects like softening and hardening be heavier [16]. Rezazadeh et al. [17] studied of parametric oscillation by using variation iteration method. Furthermore, D. Younesian et al. [18] investigated the dynamics solutions to primary and secondary resonance micro-beams.

In this paper is devoted to study subharmonic solution of even order ($\frac{1}{2}$) and ($\frac{1}{4}$) to a weakly nonlinear second order differential equation which governs the motion of MEMS.

2 Perturbation Analysis

From Eq.(6) in [19], an during Taylor expansion and retained only certain only term in the L.H.S terms, we obtain the following weakly nonlinear second order differential equation

$$\begin{aligned} &u'' + 2\epsilon\mu u' + \omega_o^2 u + \epsilon(\alpha_1 u^2 + \alpha_2 u^3) - \epsilon\alpha(2u + 3u^2 + 4u^3) \\ &+ \epsilon(2u + 3u^2 + 4u^3)(F_1 \cos(\Omega t) + F_2 \cos(2\Omega t)) \\ &+ \epsilon(\alpha + F_1 \cos(\Omega t) + F_2 \cos(2\Omega t)) = 0. \end{aligned} \tag{2.1}$$

Equation (2.1) represents modified Duffing equation subjected to a weakly nonlinear parametric and external excitations. This equation describes the main motions at time scales of the natural vibrations of the microstructure and fast dynamic at time scales of the high-frequency voltage [19], ϵ is a small parameter $\epsilon \ll 1$, μ is the coefficient of viscous damping, ω_o is the linear natural frequency, Ω is the frequency of the external excitation, α is the coefficient of linear term. α_1 and

α_2 are the coefficients of the nonlinear terms, F_1 and F_2 are constants.

By using the method of multiple scales [20, 21, 22, 23], one seeks a uniform substantial first order expansion in the form

$$u(t;\epsilon) = u_o(T_o, T_1) + \epsilon u_1(T_o, T_1) + O(\epsilon^2), \quad T_n = \epsilon^n t, \tag{2.2}$$

where the fast time scale $T_o = t$ and the slow time scale $T_1 = \epsilon t$. In terms of T_n the time derivatives become

$$\frac{d}{dt} = D_o + \epsilon D_1 + ..., \quad \frac{d^2}{dt^2} = D_o^2 + 2\epsilon D_o D_1 + ..., \tag{2.3}$$

where $D_n = \frac{\partial}{\partial T_n}$. Substituting Eqs.(2.2) and (2.3) into Eq.(2.1) and equating terms with the same power of on both sides, we get a system of linear partial differential equations

$$ZeroOrder : D_o^2 u_o + \omega_o^2 u_o = 0. \tag{2.4}$$

$$\begin{aligned} FirstOrder : D_o^2 u_1 + \omega_o^2 u_1 = & -2D_o D_1 u_o - 2\mu D_o u_o - \alpha_4 u_o^3 - \alpha_3 u_o^2 \\ & - 3\cos(2\Omega T_o) F_2 u_o^2 - \cos(\Omega T_o) F_1 \\ & - \cos(2\Omega T_o) F_2 + \alpha_1 - 2\cos(\Omega T_o) F_1 u_o \\ & - 2\cos(2\Omega T_o) F_2 u_o + \alpha_2 u_o - 3\cos(\Omega T_o) F_1 u_o^2. \end{aligned} \tag{2.5}$$

The solution of Eq.(2.4) can be expression the form

$$u_o(T_o, T_1) = A(T_1) e^{i\omega_o T_o} + \bar{A}(T_1) e^{-i\omega_o T_o}, \tag{2.6}$$

where $\bar{A}$ is the complex conjugate of A, which is an arbitrarily complex function of T_1 at this level of approximation. It is determined by imposing the solvability condition at this level of approximation. Carrying out the basic details of the method of multiple scales, we obtain a first approximation of the Eq.(2.7) as

$$u_o(T_o, T_1) = a\cos(\Omega t + \varphi) + O(\epsilon). \tag{2.7}$$

Substituting Eq.(2.6) into Eq.(2.5) yield

$$\begin{aligned} D_o^2 u_1 + \omega_o^2 u_1 = & -(-2\alpha A + 2i\mu\omega_o A - 12\alpha A^2\bar{A} + 3A^2\alpha_2\bar{A} + 2i\omega_o A')e^{i\omega_o T_o} \\ & + 6\alpha A\bar{A} - 2A\alpha_1\bar{A} + \frac{3}{2}F_1\bar{A}^2 e^{i(\Omega-2\omega_o)T_o} + \frac{3}{2}F_2\bar{A}^2 e^{i(2\Omega-2\omega_o)T_o} \\ & + (F_1\bar{A} + 6AF_1\bar{A}^2)e^{i(\Omega-\omega_o)T_o} + (F_2\bar{A} + 6AF_2\bar{A}^2)e^{i(2\Omega-\omega_o)T_o} \\ & + (\frac{F_1}{2} + 3AF_1\bar{A})e^{i\Omega T_o} + (\frac{F_2}{2} + 3AF_2\bar{A})e^{2i\Omega T_o} \\ & + 2F_1\bar{A}^3 e^{i(\Omega-3\omega_o)T_o} + 2F_2\bar{A}^3 e^{i(2\Omega-3\omega_o)T_o} + NST. + c.c, \end{aligned} \tag{2.8}$$

where NST denotes the terms which do not produce secular terms. Any particular solution of Eq.(2.8) contain the secular term, and it may include small divisor term depending on subharmonic solutions of order $(\frac{1}{2}, \frac{1}{4})$. In this paper, the case of subharmonic solutions of even order $\Omega \approx 2\omega_o$ and $\Omega \approx 4\omega_o$, are considered. The next step in the application of the method of multiple scales is to transform the small divisor term into secular term by introducing the detuning parameters σ_1 and σ_2 as follows

3 Subharmonic Solution of Order $(\frac{1}{2})$ $(\Omega \approx 2\omega_o)$

Let $\Omega \approx 2\omega_o$, we can transform the small divisor term into secular term by introduce the detuning parameter σ_1.
i.e

$$\Omega = 2\omega_o + \epsilon\sigma_1, \tag{3.1}$$

Inserting Eq.(3.1) into Eq.(2.8) and eliminating the terms that produce secular terms leads to

$$\begin{aligned}&-2i\omega_o A' - A\alpha_2 - 2i\mu A\omega_o - 3A^2\alpha_4\bar{A} + ((F_1\bar{A} + 6AF_1\bar{A}^2))e^{i\sigma_1 T_1}\\&+2F_2\bar{A}^3e^{2i\sigma_1 T_1} = 0\end{aligned} \tag{3.2}$$

In order to solve equation (3.2), we express A in the polar form

$$A(T_1) = \frac{1}{2}a(T_1)e^{i\beta(T_1)}, \tag{3.3}$$

where a and β are functions of T_1. Substituting Eq.(3.3) into Eq.(3.2) and separating real and imaginary parts yields

$$\acute{a} = -\frac{1}{2}\mu a + \frac{1}{2\omega_o}(1+3a^2)aF_1\sin\gamma_1 + \frac{1}{4\omega_o}a^3F_2\sin 2\gamma_1, \tag{3.4}$$

$$\begin{aligned}a\acute{\gamma_1} =& -a\sigma_1 - \frac{a\alpha}{2\omega_o} - \frac{3\alpha a^3}{4\omega_o} - \frac{3\alpha_2 a^3}{8\omega_o} + \frac{1}{2\omega_o}(1+3a^2)aF_1\cos\gamma_1\\&+\frac{1}{4\omega_o}a^3F_2\cos 2\gamma_1,\end{aligned} \tag{3.5}$$

where a and γ_1 respect to the amplitude and the phase.

$$\gamma_1 = \sigma_1 T_1 - 2\beta. \tag{3.6}$$

It is obvious that, Eqs.(3.4) and (3.5) have a trivial solution which of corresponds to the trivial steady state solution. Non-trivial steady state solution correspond to the non-trivial fixed points (equilibrium points) of Eqs.(3.4) and (3.5). That is, they satisfy $\acute{a} = \acute{\gamma_1} = 0$, and are given by

$$-\frac{1}{2}\mu a + \frac{1}{2\omega_o}(1+3a^2)aF_1\sin\gamma_1 + \frac{1}{4\omega_o}a^3F_2\sin 2\gamma_1 = 0, \tag{3.7}$$

$$\begin{aligned}\frac{1}{2}a\sigma_1 =& -\frac{a\alpha}{2\omega_o} - \frac{3\alpha a^3}{4\omega_o} - \frac{3\alpha_2 a^3}{8\omega_o} + \frac{1}{2\omega_o}(1+3a^2)aF_1\cos\gamma_1\\&+\frac{1}{4\omega_o}a^3F_2\cos 2\gamma_1,\end{aligned} \tag{3.8}$$

Equations (3.7) and (3.8) show that there are two possibilities: (trivial solution) at $a = 0$ and (nontrivial solution) at $a \neq 0$. Squaring (3.7), (3.8) and adding them, we get the frequency-response equation. Eliminating $\sin\gamma_1$ and $\cos\gamma_1$ from Eqs.(3.7) and (3.8) yields the frequency response equation i.e.

$$\begin{aligned}\sigma_1 =& \frac{-12a^2\alpha\omega_0 + 3a^2\alpha_2\omega_0}{4\omega_0^2}\\&\pm\frac{\sqrt{n_1F_1^2\omega_0^2 + a^4F_2^2\omega_0^2 + n_2F_1F_2\omega_0^2 - 16\mu^2\omega_0^4} - 8\alpha\omega_0}{2\omega_0^2},\end{aligned} \tag{3.9}$$

where $n_1 = 9a^4 + 12a^2 + 4$ and $n_2 = 6a^4 + 4a^2$ Now, the analysis of the stability of the trivial solutions is equivalent to the analysis of the linear solutions of equation (3.2) by neglecting the nonlinear terms, we get

$$2i\omega_o\acute{A} + A\alpha_2 + 2i\mu A\omega_o - \bar{A}F_1e^{i\sigma_1 T_1} = 0. \tag{3.10}$$

To solve Eq.(3.10) and lets $A = (B(T_1) + ib(T_1))e^{i\frac{1}{2}\sigma_1(T_1)}$, where B and b are real and imaginary parts, so we get

$$\acute{b} + \mu b + \Gamma_1 B = 0, \tag{3.11}$$

$$\acute{B} + \mu B - \Gamma_2 b = 0, \tag{3.12}$$

where $\Gamma_1 = (\frac{1}{2}\sigma_1 + \frac{\alpha}{\omega_o} + \frac{F_1}{2\omega_o})$ and $\Gamma_2 = (\frac{1}{2}\sigma_1 - \frac{\alpha}{\omega_o} + \frac{F_1}{2\omega_o})$.
Eqs.(3.11) and (3.12) admit solution of the form $(B, b) \propto (b_1, b_2)e^{\theta_o T_1}$, where (b_1, b_2) are constant. The eigenvalues of the coefficient matrix of Eqs.(3.11) and (3.12) are

$$\theta_o = -\mu \pm i\Gamma_1\Gamma_2. \tag{3.13}$$

The solution is unstable if and only if the real part of the fixed points are positive.

To determine the stability of the nontrivial solutions, we use the averaged equations (3.4) and (3.5) when the last term in these equations does not exist and let the nontrivial solutions have small variation from the steady state solutions a_o and γ_{10} so that

$$a(T_1) = a_0 + a_1(T_1) \qquad \& \qquad \gamma_1(T_1) = \gamma_{10} + \gamma_{11}(T_1), \tag{3.14}$$

where a_1 and γ_{11} are assumed to be infinitesimal. Thus, the solution of equations (3.7) and (3.8) are stable or unstable depending on whether the functions a_1 and γ_{11} decay or grow with timeT_1. Inserting equations (3.14) into equations (3.4) and (3.5) when the terms containing γ_1 in these equations does not exist and keeping only linear terms in the perturbed quantities, using steady-state equations (3.7) and (3.8),

we obtain

$$\begin{aligned} \acute{a_1} &= (-\mu + \frac{1}{2\omega_o}(1 + 9a_o^2)F_1 \sin\gamma_o + \frac{3}{4\omega_o}a_o^2 F_2 \sin 2\gamma_o)a_1 \\ &+ (\frac{1}{2\omega_o}(3 + a_o)F_1 \cos\gamma_o + \frac{1}{2\omega_o}a_o^3 F_2 \cos 2\gamma_o)\gamma_{11} \end{aligned} \tag{3.15}$$

$$\begin{aligned} \acute{\gamma_{11}} &= (\Delta + \frac{1}{a_o\omega_o}(1 + 9a_o^2)F_1 \cos\gamma_o - \frac{3}{2\omega_o}a_o F_2 \cos 2\gamma_o)a_1 \\ &+ (\frac{1}{\omega_o}(1 + 3a_o^2)F_1 \sin\gamma_o - \frac{1}{\omega_o}a_o^2 F_2 \sin 2\gamma_o)\gamma_{11}, \end{aligned} \tag{3.16}$$

where $\Delta = \frac{1}{a_o}\sigma_1 - \frac{2}{a_o\omega_o} + \frac{9\alpha}{2\omega_o}a_o + \frac{9\alpha_2}{4\omega_o}a_o$.

Substituting $a_1 = \Gamma_1 e^{\theta T_1}$ and $\gamma_{11} = \Gamma_2 e^{\theta T_1}$ into Eq.(3.15) and Eq.(3.16). We get

$$\begin{aligned} (36a_0^2F_1 - 6a_0^2F_2 + 18a_0^2\alpha + 9a_0^2\alpha_2 + 4F_1 + 4\sigma_1\omega_0 - 8)\Gamma_1 - (4a_0\theta\omega_0)\Gamma_2 &= 0 \\ (-2\omega_0(\theta + \mu))\Gamma_1 + (a_0^3F_2 + (3a_0^3 + a_0)F_1)\Gamma_2 = 0 \end{aligned} \tag{3.17}$$

For the nontrivial solution, the determinant of the coefficient matrix for Γ_1 and Γ_2 must vanish, which leads to a quadratic equation for the eigenvalue θ.

$$\begin{aligned} \theta &= -\frac{\mu}{2} \\ &\pm \frac{\sqrt{y_1F_1 + y_2F_2 + y_3F_1^2 - 6a^4F_2^2 + y_4F_1F_2 + y_5\sigma_1\omega_o + 2\mu^2\omega_o^2}\omega_o^2}{2\sqrt{2}}, \end{aligned} \tag{3.18}$$

where $y_1 = 54a^4\alpha + 27a^4\alpha_2 + 18a^2\alpha + 9a^2\alpha_2 - 24a^2 - 8$,
$y_2 = 18a^4F_2\alpha + 9a^4F_2\alpha_2 - 8a^2$, $y_3 = 108a^4 + 48a^2 + 4$,
$y_4 = 18a^4 - 2a^2$ and $y_5 = 12a^2F_1\sigma_1\omega_o + 4a^2F_2\sigma_1\omega_o + 4F_1\sigma_1\omega_o$.

Consequently, a solution stable if and only if the real parts of both eigenvalues (3.18) are less than or equal to zero.

4 Subharmonic Solution of Order $(\frac{1}{4})$ $(\Omega \approx 4\omega_o)$

Let $\Omega \approx 4\omega_o$, we can transform the small divisor term into secular term by in introduce the detuning parameter to σ_2.
i.e

$$\Omega = 4\omega_o + \epsilon\sigma_2 \tag{4.1}$$

Introduce Eqs.(4.1) into Eqs.(2.8), eliminating the terms, which produce the secular terms, we get

$$-2i\omega_o A' - A\alpha_2 - 2i\mu A\omega_o - 3A^2\alpha_4\bar{A} + F_1\bar{A} + 2F_1\bar{A}^3 e^{i\sigma_2 T_1} = 0. \tag{4.2}$$

In order to solve equation (4.2), we express A in the polar form

$$A(T_1) = \frac{1}{2}a(T_1)e^{i\beta(T_1)}, \tag{4.3}$$

where a and β are functions of T_1. Substituting Eq.(4.3) into Eq.(4.2) and separating real and imaginary parts yields

$$\acute{a} = -\frac{1}{2}\mu a + \frac{1}{8\omega_o}a^3 F_1 \sin\gamma_2, \tag{4.4}$$

$$a\acute{\gamma_2} = -(\frac{1}{4}a\sigma_2 + \frac{a\alpha}{2\omega_o} + \frac{3\alpha a^3}{4\omega_o} + \frac{3\alpha_2 a^3}{8\omega_o}) + \frac{1}{8\omega_o}a^3 F_1 \cos\gamma_2, \tag{4.5}$$

where a and γ_2 respect to the amplitude and the phase.

$$\gamma_2 = \sigma_2 T_1 - 4\beta. \tag{4.6}$$

It is obvious that, Eqs.(4.4) and (4.5) have a trivial solution which correspond to the trivial steady state solution. Non-trivial steady state solution correspond to the non-trivial fixed points (equilibrium points) of Eqs.(4.4) and (4.5). That is, they satisfy $\acute{a} = \acute{\gamma_1} = 0$, and are given by

$$\frac{1}{8\omega_o}a^3 F_1 \sin\gamma_2 = \frac{1}{2}\mu a \tag{4.7}$$

$$\frac{1}{8\omega_o}a^3 F_1 \cos\gamma_2 = \frac{1}{4}a\sigma_2 + \frac{a\alpha}{2\omega_o} + \frac{3\alpha a^3}{4\omega_o} + \frac{3\alpha_2 a^3}{8\omega_o} \tag{4.8}$$

Eliminating $\sin\gamma_2$ and $\cos\gamma_2$ from Eqs.(4.7) and (4.8) yields the frequency-response equation

$$\begin{aligned} &64a^2\alpha^2 + 192a^4\alpha^2 + 144a^6\alpha^2 - 4a^6F_1^2 - 48a^4\alpha\alpha_2 + 9a^6\alpha_2^2 + 4a^2\sigma_2^2\omega_0^2 \\ &- 72a^6\alpha\alpha_2 + 32a^2\alpha\sigma_2\omega_0 + 48a^4\alpha\sigma_2\omega_0 - 12a^4\sigma_2\alpha_2\omega_0 + 64a^2\mu^2\omega_0^2 = 0, \end{aligned} \tag{4.9}$$

i.e.

$$\sigma_2 = \frac{1}{2\omega_o^2}(\kappa_1 + \kappa_2 a^2 \pm 2\sqrt{a^4F_1^2\omega_o^2 - 16\mu^2\omega_o^4}), \tag{4.10}$$

where $\kappa_1 = -8\alpha\omega_o, \kappa_2 = -12\alpha\omega_o + 3\alpha_2\omega_o$.

To determine the stability of the nontrivial steady state solutions given by Eqs.(4.7) and (4.8). To derive the stability criteria, we need to examine the behavior of a small deviation from the steady-state solutions a_o and γ_{20}. Thus, we assume that

$$a(T_1) = a_0 + a_1(T_1) \qquad \gamma_2(T_1) = \gamma_{20} + \gamma_{21}(T_1). \tag{4.11}$$

Where a_1 and γ_{21} are assumed to be infinitesimal. Inserting Eq.(4.11) into Eqs.(4.4) and (4.5) when the terms containing in these equations does not exist and using the steady-state equations (4.7) and (4.8), we get

$$\acute{a_1} = (2\mu)a_1 - (\frac{8a_o\alpha - 12a_o^3\alpha + 3a_o^3\alpha_2 - 2a_o\sigma_2\omega_o}{8\omega_o})\gamma_{21}. \tag{4.12}$$

$$\acute{\gamma_{21}} = -2(\frac{4\alpha + \sigma_2\omega_o}{a_o\omega_o})a_1 + (2\mu)\gamma_{21} \tag{4.13}$$

Equations (4.12) and (4.13) admit solution of the form $(a_1, \phi_1) \propto (c_1, c_2)e^{\theta T_1}$, where (c_1, c_2) are constants. Provided that,

$$\theta = -\mu \pm \frac{1}{8\omega_o^2}\sqrt{64\mu^2\omega_o^4 + 16\omega_o^2(16\kappa_3\alpha^2 + \kappa_4\alpha\alpha_2 + \kappa_5\alpha\sigma_2\omega_o + (32\mu^2 + 2\sigma_2^2)\omega_o^2)}, \tag{4.14}$$

where $\kappa_3 = 32 + 48a_o^2$, $\kappa_4 = -12a_o^2$ and $\kappa_5 = 16 + 12a_o^2$. The solution is unstable if and only if the real part of the fixed points are positive.

5 Numerical Results and Discussion

The frequency response equations (3.9) and (4.9) are nonlinear algebraic equations, which are solved numerically. The numerical results are shown in Figs. (1-15). The stable and unstable solutions are represented by solid and dashed lines respectively on the response curves. Figs. (1-7) represent the frequency response curves of the subharmonic solution of order (one-to-two) for the parameters

$(\omega_0 = 2, \mu = .04, \alpha_2 = 1, \alpha = 0.3, F_1 = 0.05, F_2 = 5)$ and Figs (8-15) represent the frequency response curves of the subharmonic solution of order (one-to-four) for the parameters $(\omega_0 = 1, \mu = 0.02, \alpha_2 = 4, \alpha = 0.3, F_1 = 3)$. In all Figs. (1-15) the trivial solution is stable.

In Figs.(1,2) we note that the response amplitude has a single valued curve so that the minimum value exist at the point $\sigma_1 = 0.244$ and has consisted of two stable branches. For increasing α, we observe that the right branch shifts to the left and the left branch shifts downwards. The multivalued curve has decreased magnitudes and the minimum value shifts to the left. There exist a saddle-node bifurcation in the upper branch at the point $\sigma_1 = 2.11$. For further increasing in α we note that the multivalued curve moves downwards and has decreased magnitudes, and the saddle-node bifurcation exists at the point $\sigma_1 = 2.044$. For increasing the coefficients of linear and nonlinear parametric excitations F_1, we noted that there exist discontinuities in the single-valued curve and separated into two branches so that the two branches shifts down inwards and have decreased magnitudes respectively, Fig.(3). As the coefficients of linear and nonlinear parametric excitations $F_2 = 3$, we observe that the single-valued curve shifts upwards and has the same minimum value. For further increasing of F_2, the single-valued curve contracted so that the zone of definition and stability are decreased and there exist a saddle-node bifurcation at $\sigma_1 = 1.69$, Fig.(4). When the damping factor μ takes the values (0.9, 1.5), we note that the single-valued curve shifts upwards and the minimum value has increased magnitudes respectively, Fig.(5). For decreasing and increasing the natural frequency ω_o, we observe that the single-valued curve shifts downwards and upwards so that the minimum value has decreased and increased magnitudes respectively, Fig.(6). As the coefficient of cubic nonlinearity $\alpha_2 = 3$ is increased, we note that the left and right branches of the single-valued curve shifts upwards and downwards so that it have increased and decreased magnitudes respectively. The region of definition and stability are decreased, and there exist a saddle-node bifurcation at $\sigma_1 = 0.722$. When α_2 decreases, the left and right branches of the single-valued curve shifts downwards and upwards so that it have decreased and increased magnitudes respectively, Fig.(7).

Figs.(8-15) represent the frequency-response curves of subharmonic oscillation of order (one-to-fourth). In Figs.(8,9), we observe that the response amplitude has multivalued curve and consists of two branches so that the upper and down branches have stable and unstable solutions respectively. There exist a saddle-node bifurcation at the point $\sigma_2 = 0.19$. When α takes the values (0.1 and 0.001), we note that the multivalued curve shifts downwards and has decreased magnitudes respectively. The saddle-node bifurcations exist at the points $\sigma_2 = 0.17$ and $\sigma_2 = 0.18$, Fig.(10). As $\alpha_2 = 9$, we observe that that the multivalued curve contracted and crossed the main multivalued curve and with decreased magnitudes. There exist a saddle-node bifurcation. When α_2 is increased further, the multivalued curve shifts downwards and has decreased magnitudes and the saddle-node bifurcation exist at the point $\sigma_2 = 0.164$, Fig.(11). For increasing $F_1 = 0.9, 3$ respectively, we note that the multivalued curve contracted and contained in the main multivalued curve so that the upper and down branches have decreased and increased magnitudes. The zones of multivalued, stability and definition are decreased. The saddle-node bifurcations exist at the points $\sigma_2 = 0.30$ and $\sigma_2 = 0.20$, Fig.(12). When ω_o takes the values (1 and 0.3), we observe that the multivalued curve shifts to the left and move downwards and intersect each other respectively. The regions of definition, multivalued and stability are increased. The saddle-node bifurcations exist at the points $\sigma_2 = 0.19$ and $\sigma_2 = 0.11$, Fig.(13). For increasing the damping factor $\mu = 0.02, 0.4$, we note that the multivalued curve contracted and shifts upwards so that the zones of multivalued, stability and definition are decreased. The saddle-node bifurcations exist at the points $\sigma_2 = 0.20$ and $\sigma_2 = 0.89$, Fig.(14).

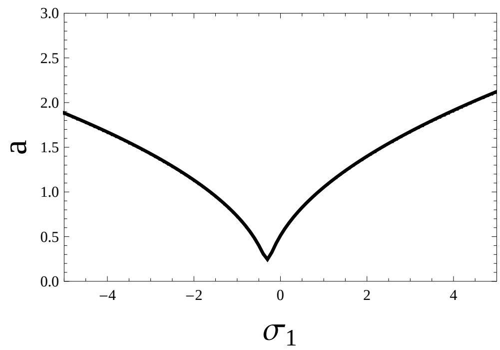

Fig. 1. The frequency response curves of the subharmonic solution of order $\frac{1}{2}$ for the parameters $\omega_o = 1, \mu = 0.02, F_1 = 3, \alpha = 0.3, \alpha_2 = 4$

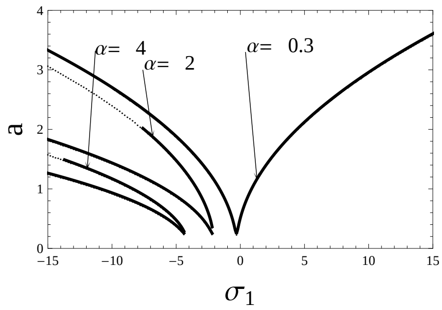

Fig. 2. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α

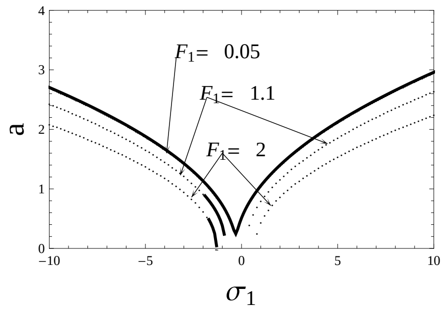

Fig. 3. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_1

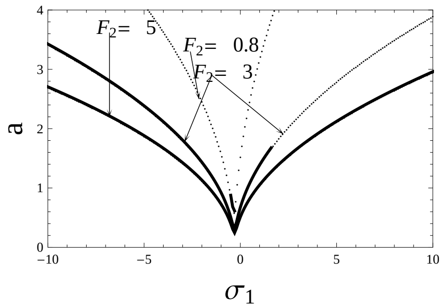

Fig. 4. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_2

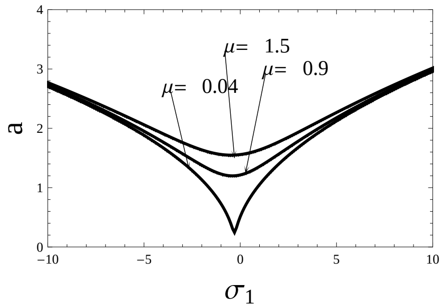

Fig. 5. The frequency response curves for different values of μ

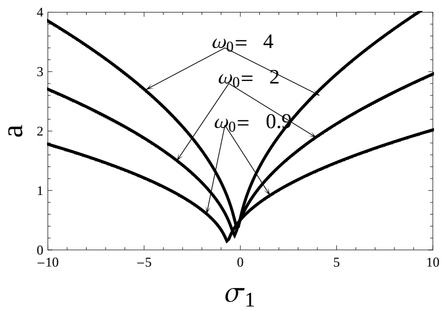

Fig. 6. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ω_o

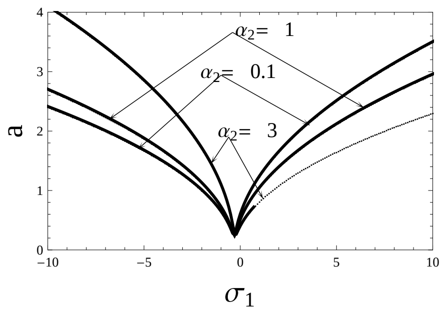

Fig. 7. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α_2

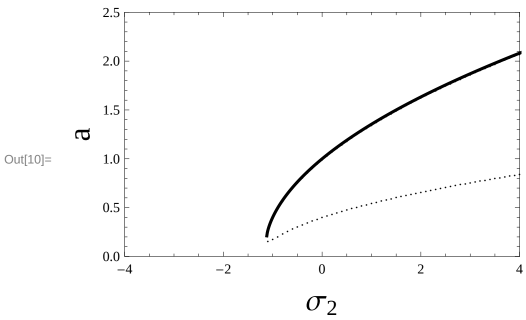

Fig. 8. The frequency response curves of the subharmonic solution of order $\frac{1}{4}$ for the parameters $\omega_o = 1, \mu = 0.02, F_1 = 3, \alpha = 0.3, \alpha_2 = 4$

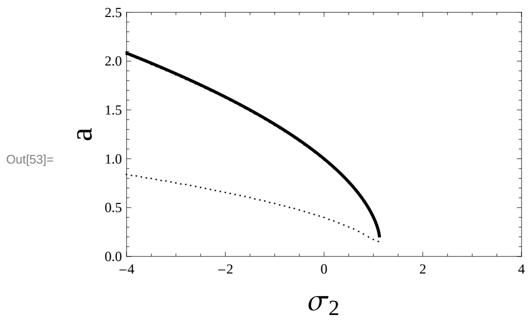

Fig. 9. The frequency response curves of the subharmonic solution of order $\frac{1}{4}$ for the parameters $\omega_o = 1, \mu = 0.02, F_1 = -3, \alpha = -0.3, \alpha_2 = -4$

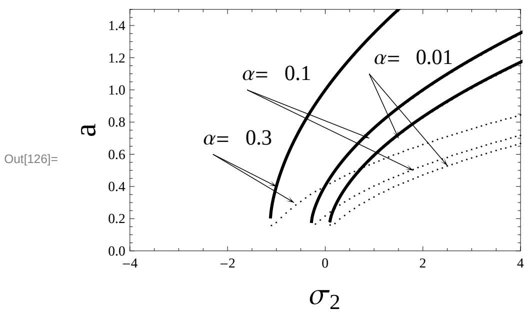

Fig. 10. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α

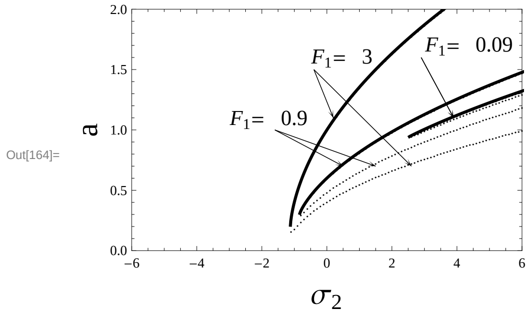

Fig. 11. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_1

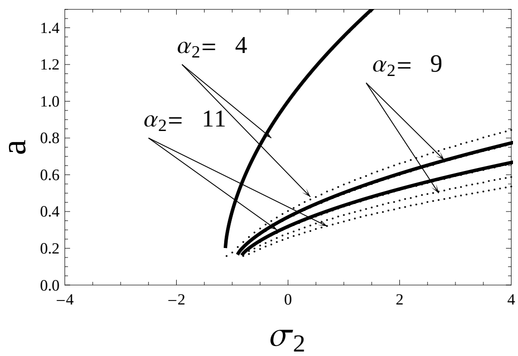

Fig. 12. Variation of the amplitude of the response with the detuning parameter for increasing α_2

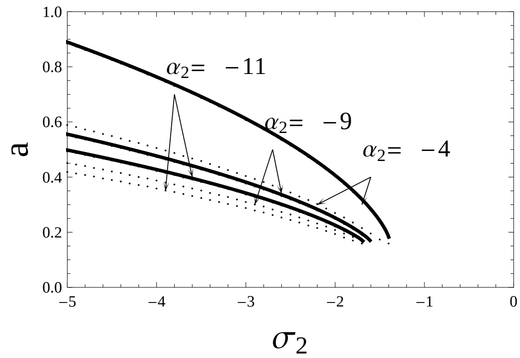

Fig. 13. Variation of the amplitude of the response with the detuning parameter for decreasing α_2

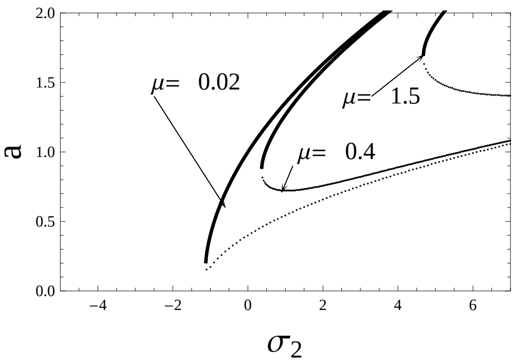

Fig. 14. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing μ

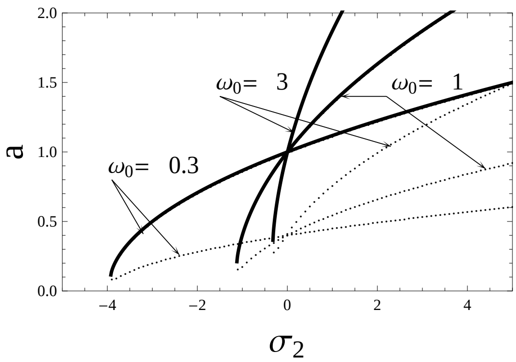

Fig. 15. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ω_o

6 Conclusion

In this paper, we study subharmonic periodic solutions of even order $(\frac{1}{2}, \frac{1}{4})$ of a weakly nonlinear second order differential equation which governed the motion of a micro-electro-mechanical systems (Micro-dynamical system) analytically by using a perturbation technique (Multiple scales method). By applying this method, we obtain the modulation equations in the amplitude and the phase. Determined the steady-state solutions (fixed-points of the modulation equations)(or equilibrium points of the micro-dynamical systems) and the frequency-response equations. Currying the stability analysis of the trivial and non-trivial solutions. Numerical results are presented graphically in group of figures, in which dashed (undashed) curves represent stable (unstable) solutions. Finally, discussion of the curves. Subharmonic of order one-to-four note that:

- The region of stability does not affect for increasing the parameters ω_o, μ and for decreasing the parameters α, μ and α_2.
- The single-valued is separated into two discontinuous branches when the parameter F_1 increases.
- For increasing α, we observe that the right branch of the single-valued curve shifts to the left and given a multivalued curve.

Subharmonic of order one-to-four note that:

- The multivalued curve contracted and lay inside the main multivalued curve for decreasing and increasing F_1 and μ respectively.
- The multivalued curve crossed the main multivalued curve and moved downwards when α takes the values 9 and 11.
- The regions of multivalued, definition and stability are increased for decreasing ω_o.

Competing Interests

Authors have declared that no competing interests exist.

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