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The Condition Numbers of Semi-simple Eigenvalue of Quadratic Eigenvalue Problem

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Authors' contributions

This work was carried out in collaboration between both authors. Author DZ designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author XH managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

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Short Research Article

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Abstract

The condition numbers of eigenvalues of matrices measure the sensitivities of eigenvalues to small perturbation of matrix. They're widely used to assess the quality of numerical algorithms for eigenvalue problems. This paper considers the condition number of multiple eigenvalue of regular quadratic eigenvalue problem. Based on the properties of multiple eigenvalue of quadratic eigenvalue problem analytically dependent on several parameters, we give various definitions for condition numbers of semi-simple eigenvalue of regular quadratic eigenvalue problem. Utilizing SVD and the properties of unitarily invariant norm, we derive the computational expressions for the introduced condition numbers. We find that the condition numbers defined can be computed in terms of the singular values of $X_1Y_1^T$, where $X_1Y_1^T$ are respectively the right eigenvector matrix and left eigenvector matrix corresponding to the multiple eigenvalue. Compared with the existing condition numbers of multiple eigenvalues of quadratic eigenvalue problem, the condition numbers defined in this paper can measure not only the worst case sensitivity of semi-simple eigenvalue, but also the different sensitivities of the eigenvalues spawned from semi-simple eigenvalue.

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1 Introduction

The condition numbers of eigenvalues of matrices measure the sensitivities of eigenvalues to small perturbations of the matrices. They are important to assess the quality of numerical algorithms for eigenvalue problems. For linear eigenvalue problems, there have been many results [1,2,3,4]. However, in many fields such as stability analysis of damped dynamic systems, eigenstructure assignment of control design systems [5], it's essentially important to consider the condition numbers of eigenvalues of the following quadratic eigenvalue problem (QEP).

$$\left(\lambda^2 M + \lambda C + K\right) x = 0, \tag{1.1}$$

where $M, C, K \in C^{n\times n}$.

[5,6] considered the condition number of simple eigenvalue of QEP (1.1). [7,8] investigated the condition number of simple eigenvalue of matrix polynomials

$$P(\lambda) = \sum_{j=0}^{m} A_j \lambda^j . \tag{1.2}$$

However, all aforementioned work requires that the leading coefficient matrices M, A_m are invertible, which excludes the case of infinite eigenvalues. [9] removed this restriction and studied the condition number of simple eigenvalue of homogeneous matrix polynomials

$$L(\alpha,\beta) = \sum_{j=0}^{m} A_j \alpha^j \beta^{m-j} , \tag{1.3}$$

and it allows infinite eigenvalue.

There is little research about the condition number of multiple eigenvalue of quadratic eigenvalue problem and polynomial eigenvalue problem. In [8], the condition number of multiple eigenvalue of matrix polynomial (1.2) is introduced, and the relationship between the introduced condition number and pseudospectral growth rate is revealed. [10] defined the condition number of multiple eigenvalue of (1.3) via the Bauer-Fike Theorem. The condition numbers defined in [8,10] only can measure the worst case perturbation of multiple eigenvalue. However, the multiple eigenvalue λ with multiplicity r often split into r simple eigenvalues when perturbed. So it is natural to have r condition numbers for multiple eigenvalue λ.

In this paper, based on the directional derivatives of semi-simple eigenvalue[1] of QEP analytically dependent on several parameters, we define and analyze the condition number of semi-simple eigenvalue of quadratic eigenvalue problem (1.1). We prove that the defined condition number can be computed in terms of the singular values of $X_1 Y_1^T$, where the columns of $X_1, Y_1 \in C_r^{n\times r}$ are respectively the right and left eigenvectors corresponding to multiple eigenvalue λ. The introduced condition number can measure not only the worst case perturbation of the semi-simple eigenvalue, but also the corresponding sensitivities of different eigenvalues spawned from the semi-simple eigenvalue.

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[1]*Let λ be an eigenvalue of (1.1). If the algebraic multiplicity of λ is greater than one, and it is equal to the geometric multiplicity of λ, then λ is called the semi-simple eigenvalue of (1.1).*

The rest of the paper are organized as follows. In section 2, we introduce some basic concepts and results concerning quadratic eigenvalue problem. In section 3, we define the condition numbers of semi-simple eigenvalue of regular quadratic eigenvalue problem. Their computational expressions and bounds are derived. Moreover, we give an example to test our conclusions.

For convenience, we use the following notations. $C_r^{m\times n}$ denotes the set of all complex $m\times n$ matrices of rank r. A^T denotes the transpose of matrix A. A^H denotes the conjugate transpose of matrix A. $\|A\|_F$ and $\|A\|_2$ respectively stand for the Frobenius norm and spectral norm of matrix A. $\rho(A)$ is the spectral radius of matrix A. $\lambda_1(A),...,\lambda_n(A)$ denote the eigenvalues of $n\times n$ matrix A. $\bar{x}$ denotes the conjugate of complex number x.

2 Theoretical Base

In this section, we introduce some necessary definitions and conclusions, which are the basis for our studying the condition number of semi-simple eigenvalue of QEP (1.1). For brevity, eigenvalues and eigenvectors of (1.1) are also called eigenvalues and eigenvectors of matrix triple $\{M, C, K\}$.

Now consider the following quadratic eigenvalue problem:

$$[\lambda^2(p)M(p)+\lambda(p)C(p)+K(p)]x(p)=0, \lambda(p)\in C, x(p)\in C^n$$

where $p=(p_1,...,p_N)^T\in C^N$, $M(p),C(p),K(p)\in C^{n\times n}$ are analytic on a neighborhood $N(p^*)$ of p^*.

Let λ_1 be a semi-simple eigenvalue of $\{M(p^*),C(p^*),K(p^*)\}$ with multiplicity r. Theorem 2.2 and Theorem 3.1 in [11] gave the directional derivative[2] of the eigenvalues splitted from λ_1. For convenience of our discuss, we recite the results in Theorem 2.2 and Theorem 3.1 of [11] as the following theorem.

Theorem 2.1: Let $M(p),C(p),K(p)\in C^{n\times n}$ be analytic on a neighborhood $N(p^*)$ of $p^*\in C^N$. If λ_1 is a semi-simple eigenvalue of $\{M(p^*),C(p^*),K(p^*)\}$ with multiplicity r, i.e., there exist matrices $X_1,Y_1\in C_r^{n\times r}$ such that the columns of X_1,Y_1 are respectively the right and the left eigenvectors corresponding to λ_1, and $Y_1^T\left(2\lambda_1 M(p^*)+C(p^*)\right)X_1=I_r$, then

(1) There exists a neighborhood $N_1(p^*)\subseteq N(p^*)$ of p^* and r functions $\lambda_1(p),...,\lambda_r(p)$, such that $\lambda_1(p),...,\lambda_r(p)$ are the eigenvalues of $\{M(p),C(p),K(p)\}$, and $\lambda_i(p)(i=1,...,r)$ are continuous at p^*, and $\lambda_i(p^*)=\lambda_1(i=1,...,r)$;

[2] *Let N be an open set of C^N, and $p^*\in N$, and $f(p)$ be a function defined on N, and $v\in C^N$ with $\|v\|_2=1$. If $\lim\limits_{t\to 0^+}\frac{f(p^*+tv)-f(p^*)}{t}$ exist, then the limit value is called the directional derivative of $f(p)$ in the direction v at p^*, denoted by $D_v f(p^*)$.*

(2) For any fixed direction $v=(v_1,\ldots,v_N)^T \in C^N$ with $\|v\|_2=1$, there exist $\beta>0$ and r single-valued continuous functions $\mu_1(p^*+tv),\ldots,\mu_r(p^*+tv)$ defined on $[-\beta,\beta]$, such that

(i) $\mu_1(p^*+tv),\ldots,\mu_r(p^*+tv)$ are r eigenvalues of $\{M(p^*+tv),C(p^*+tv),K(p^*+tv)\}$;

(ii) $\{\mu_i(p^*+tv)\}_{i=1}^r=\{\lambda_i(p^*+tv)\}_{i=1}^r$ for each $t\in[-\beta,\beta]$, and there is an one-to-one correspondence between the elements of set $\{\mu_i(p^*+tv)\}_{i=1}^r$ and set $\{\lambda_i(p^*+tv)\}_{i=1}^r$;

(iii) There exist a permutation π of $\{1,\ldots,r\}$ dependent on v such that

$$D_v\mu_i(p^*)=-\lambda_{\pi(i)}\left(\sum_{j=1}^N v_j Y_1^T S_j(p^*,\lambda_1)X_1\right), i=1,\ldots,r, \tag{2.1}$$

where $S_j(p^*,\lambda_1)=\lambda_1^2\dfrac{\partial M(p^*)}{\partial p_j}+\lambda_1\dfrac{\partial C(p^*)}{\partial p_j}+\dfrac{\partial K(p^*)}{\partial p_j}$.

By Theorem 2.1, we have the following result on the sensitivity of λ_1 in the worst case.

Theorem 2.2. Under the conditions of Theorem 2.1, define

$$s_v(\lambda_1)=\lim_{\substack{t\in R\\ t\to 0}}\frac{\max_{1\le k\le r}\left|\lambda_k(p^*+tv)-\lambda_1\right|}{|t|}, \tag{2.2}$$

Then

$$s_v(\lambda_1)=\rho\left(\sum_{j=1}^N v_j Y_1^T S_j(p^*,\lambda_1)X_1\right). \tag{2.3}$$

Proof. By Theorem 2.1, for any fixed direction $v\in C^N$ with $\|v\|_2=1$, there exist $\beta>0$ and r single-valued continuous functions $\mu_1(p^*+tv),\ldots,\mu_r(p^*+tv)$ defined on $[-\beta,\beta]$, such that $\{\mu_i(p^*+tv)\}_{i=1}^r=\{\lambda_i(p^*+tv)\}_{i=1}^r$ for each $t\in[-\beta,\beta]$. Hence, $\max_{1\le k\le r}\left|\lambda_k(p^*+tv)-\lambda_1\right|=\max_{1\le k\le r}\left|\mu_k(p^*+tv)-\lambda_1\right|$ ($t\in[-\beta,\beta]$). Further from (2.2), we have

$$s_v(\lambda_1)=\lim_{\substack{t\in R\\ t\to 0}}\frac{\max_{1\le k\le r}\left|\mu_k(p^*+tv)-\lambda_1\right|}{|t|}. \tag{2.4}$$

From (2.1) we have

$$\lim_{\substack{t\in R\\ t\to 0+}} \frac{\max_{1\le k\le r}\left|\mu_k(p^*+tv)-\lambda_1\right|}{|t|} = \max_{1\le k\le r}\left|D_v\mu_k(p^*)\right| = \rho\left(\sum_{j=1}^{N} v_j Y_1^T S_j(p^*,\lambda_1)X_1\right). \qquad (2.5)$$

Observe that

$$\lim_{\substack{t\in R\\ t\to 0-}} \frac{\mu_k(p^*+tv)-\lambda_1}{t} = -D_{-v}\mu_k(p^*).$$

Further from (2.1), we have

$$\lim_{\substack{t\in R\\ t\to 0-}} \frac{\max_{1\le k\le r}\left|\mu_k(p^*+tv)-\lambda_1\right|}{|t|} = \max_{1\le k\le r}\left|D_{-v}\mu_k(p^*)\right| = \rho\left(\sum_{j=1}^{N} v_j Y_1^T S_j(p^*,\lambda_1)X_1\right). \qquad (2.6)$$

Combining (2.5) with (2.6) we get (2.3).\#

3 Condition Numbers of Semi-simple Eigenvalue

Throughout this section, we assume that

(1) $\lambda_1 \in C$ is a semi-simple eigenvalue of (1.1) with multiplicity r;

(2) The columns of $X_1, Y_1 \in C_r^{n\times r}$ are respectively the right eigenvectors and the left eigenvectors of (1.1) corresponding to λ_1, and $Y_1^T(2\lambda_1 M + C)X_1 = I_r$.

Based on Theorem 2.1 and Theorem 2.2, we will define the condition numbers of λ_1, and derive their computational expressions.

Let $E,F,G \in C^{n\times n}$ with $\left\|[E,F,G]\right\|_F = 1$ and consider matrix triple

$$\{M(t),C(t),K(t)\} = \{M+tE, C+tF, K+tG\}. \qquad (3.1)$$

By Theorem 2.1, there exist r single-valued continuous functions $\mu_1(t),...,\mu_r(t)$ such that $\mu_1(t),...,\mu_r(t)$ are the eigenvalues of (3.1), and

$$\begin{aligned}\mu_i(t) &= \lambda_1 + D_{[E,F,G]}\mu_i(0)t + o(t)\\ &= \lambda_1 - \lambda_{\pi(i)}\left(Y_1^T(\lambda_1^2 E + \lambda_1 F + G)X_1\right)t + o(t), \quad t\to 0, \quad i = 1,...,r\end{aligned} \qquad (3.2)$$

Further by Theorem 2.2, we have

$$s_{[E,F,G]}(\lambda_1) = \rho\left(Y_1^T(\lambda_1^2 E + \lambda_1 F + G)X_1\right).$$

This means that $\rho\left(Y_1^T(\lambda_1^2E+\lambda_1F+G)X_1\right)$ reflects the sensitivity of multiple eigenvalue λ_1 when $\{M,C,K\}$ is slightly perturbed in the direction $[E,F,G]$. Thus, we can give the following definition.

Definition 3.1. Let $E,F,G\in C^{n\times n}$ with $\left\|[E,F,G]\right\|_F=1$, then

$$d\left([M,C,K],[E,F,G],\lambda_1\right)=\rho\left(Y_1^T(\lambda_1^2E+\lambda_1F+G)X_1\right)$$

is called the condition number of eigenvalue λ_1 in the direction $[E,F,G]$.

Generally the condition number of eigenvalue reflects the "worst case" sensitivity of eigenvalue with respect to small perturbations of matrix. Hence, we give the following definition.

Definition 3.2: Let

$$c\left([M,C,K],\lambda_1\right)=\sup_{\substack{E,F,G\in C^{n\times n}\\ \left\|[E,F,G]\right\|_F=1}}\rho\left(Y_1^T(\lambda_1^2E+\lambda_1F+G)X_1\right).$$

Then $c\left([M,C,K],\lambda_1\right)$ is called the condition number of eigenvalue λ_1.

Observe that $rank(X_1)=rank(Y_1)=r$. We may assume that $X_1Y_1^T$ has singular value decomposition

$$X_1Y_1^T=Pdiag(\Sigma,O)Q^H, \tag{3.3}$$

where P and Q are $n\times n$ unitary matrices, $\Sigma=diag(\sigma_1,...,\sigma_r)$ with $\sigma_1\geq\cdots\geq\sigma_r>0$. Now we utilize (3.3) to give the computable expression of the quantity $c\left([M,C,K],\lambda_1\right)$.

Theorem 3.1. $c([M,C,K],\lambda_1)=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\left\|X_1Y_1^T\right\|_2$.

Proof. For any $E,F,G\in C^{n\times n}$ with $\left\|[E,F,G]\right\|_F=1$, we have

$$\begin{aligned}\rho\left(Y_1^T(\lambda_1^2E+\lambda_1F+G)X_1\right)&=\rho\left(X_1Y_1^T(\lambda_1^2E+\lambda_1F+G)\right)\\ &\leq\left\|X_1Y_1^T(\lambda_1^2E+\lambda_1F+G)\right\|_F\\ &\leq\left\|X_1Y_1^T\right\|_2\left\|\lambda_1^2E+\lambda_1F+G\right\|_F\\ &\leq\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\left\|X_1Y_1^T\right\|_2.\end{aligned}$$

Then,

$$c\left([M,C,K],\lambda_1\right)\leq\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\left\|X_1Y_1^T\right\|_2; \tag{3.4}$$

Suppose that $X_1Y_1^T$ has singular value decomposition (3.3), and $P=[p_1,\hat{P}]$, $Q=[q_1,\hat{Q}]$, where $p_1,q_1\in C^n$. Take

$$E=\frac{\overline{\lambda_1^2}}{\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}}q_1p_1^H,\ F=\frac{\overline{\lambda_1}}{\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}}q_1p_1^H,\ G=\frac{1}{\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}}q_1p_1^H.$$

Then $\left\|[E,F,G]\right\|_F=1$, and

$$\begin{aligned}\rho\left(Y_1^T(\lambda_1^2E+\lambda_1F+G)X_1\right)&=\rho\left(Y_1^T\left(\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\ q_1p_1^H\right)X_1\right)\\&=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,\rho(p_1^HX_1Y_1^Tq_1)\\&=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,\rho\left(p_1^HPdiag(\Sigma,O)Q^Hq_1\right)\\&=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,\sigma_1\\&=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,\left\|X_1Y_1^T\right\|_2.\end{aligned}\tag{3.5}$$

Hence,

$$c([M,C,K],\lambda_1)\geq\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\left\|X_1Y_1^T\right\|_2.\tag{3.6}$$

By (3.4) and (3.6), we get the result of this theorem. \#

The condition $\left\|[E,F,G]\right\|_F=1$ in Definition 3.2 is not necessary and it may be replaced by any unitarily invariant norm. So we give the following definition.

Definition 3.3: Let $\|\ \|$ be any unitarily invariant norm. Then

$$\tilde{c}([M,C,K],\lambda_1)=\sup_{\substack{E,F,G\in C^{n\times n}\\ \|[E,F,G]\|=1}}\rho(Y_1^T(\lambda_1^2E+\lambda_1F+G)X_1)$$

is called the condition number of eigenvalue λ_1.

Using the properties of unitarily invariant norm [12], we have the following result.

Theorem 3.2. Let $\tilde{c}([M,C,K],\lambda_1)$ be as in Definition 3.3. Then

$$\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\left\|X_1Y_1^T\right\|_2\leq\tilde{c}([M,C,K],\lambda_1)\leq\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\left\|X_1Y_1^T\right\|.$$

Proof. Let $X_1Y_1^T$ has singular value decomposition (3.3) and $P=[p_1,\hat{P}]$, $Q=[q_1,\hat{Q}]$, where $p_1,q_1\in C^n$. Take

$$E = \frac{\overline{\lambda_1^2}}{\sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}} q_1 p_1^H, F = \frac{\overline{\lambda_1}}{\sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}} q_1 p_1^H, G = \frac{1}{\sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}} q_1 p_1^H.$$

Observe that any unitarily invariant norm of a matrix with rank one is equal to its spectral norm [12]. Then we have

$$\begin{aligned}\|[E,F,G]\| &= \frac{1}{\sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}} \left\| \left[\overline{\lambda_1^2} q_1 p_1^H, \overline{\lambda_1} q_1 p_1^H, q_1 p_1^H \right] \right\| \\ &= \frac{1}{\sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}} \left\| \left[\overline{\lambda_1^2} q_1 p_1^H, \overline{\lambda_1} q_1 p_1^H, q_1 p_1^H \right] \right\|_2 \\ &= 1.\end{aligned}$$

Further from (3.5), we have

$$\rho\left(Y_1^T(\lambda_1^2 E + \lambda_1 F + G)X_1\right) = \sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}\, \|X_1 Y_1^T\|_2.$$

Then

$$\tilde{c}\left([M,C,K],\lambda_1\right) \geq \sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}\, \|X_1 Y_1^T\|_2. \tag{3.7}$$

For any $E,F,G \in C^{n\times n}$ with $\|[E,F,G]\| = 1$, we have

$$\rho\left(Y_1^T(\lambda_1^2 E + \lambda_1 F + G)X_1\right) \leq \|X_1 Y_1^T(\lambda_1^2 E + \lambda_1 F + G)\| \leq \|X_1 Y_1^T\| \|\lambda_1^2 E + \lambda_1 F + G\|,$$

Further from [12], we have

$$\begin{aligned}\|\lambda_1^2 E + \lambda_1 F + G\| &\leq \left\| \left[\lambda_1^2 E + \lambda_1 F + G \quad O \quad O \right] \right\| \\ &= \left\| [E,F,G] \begin{bmatrix} \lambda_1^2 I_n & O & O \\ \lambda_1 I_n & O & O \\ I_n & O & O \end{bmatrix} \right\| \\ &\leq \|[E,F,G]\| \left\| \begin{bmatrix} \lambda_1^2 I_n & O & O \\ \lambda_1 I_n & O & O \\ I_n & O & O \end{bmatrix} \right\|_2 \\ &= \sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}.\end{aligned} \tag{3.8}$$

Hence,

$$\rho\left(Y_1^T(\lambda_1^2 E + \lambda_1 F + G)X_1\right) \leq \sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}\, \|X_1 Y_1^T\|.$$

Hence,

$$\tilde{c}([M,C,K],\lambda_1) \le \sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}\,\|X_1 Y_1^T\|. \tag{3.9}$$

Combining (3.7) with (3.9), we obtain the result.\#

Specially, if we take the norm in Definition 3.3 as the spectral norm, the corresponding condition number is denoted by $\tilde{c}^{(2)}([M,C,K],\lambda_1)$. It is easily seen from Theorem 3.2 that

$$\tilde{c}^{(2)}([M,C,K],\lambda_1) = \sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}\,\|X_1 Y_1^T\|_2 .$$

By Theorem 3.2, we can introduce the following definition.

Definition 3.4: Let $\|\ \|$ be any unitarily invariant norm, then

$$\hat{c}([M,C,K],\lambda_1) = \sqrt{|\lambda_1|^4 + |\lambda_1|^2 + 1}\,\|X_1 Y_1^T\|$$

is called the condition number of eigenvalue λ_1.

When we take spectral norm and Frobenius norm in Definition 3.4, the corresponding condition numbers are respectively denoted by $\hat{c}^{(2)}([M,C,K],\lambda_1)$ and $\hat{c}^{(F)}([M,C,K],\lambda_1)$. Clearly,

$$\hat{c}^{(2)}\left([M,C,K],\lambda_1\right) = \tilde{c}^{(2)}\left([M,C,K],\lambda_1\right) = c\left([M,C,K],\lambda_1\right).$$

Above condition numbers only reflect the sensitivity of semi-simple eigenvalue of QEP (1.1) in the worst case. Nevertheless, [2] shows that it is reasonable for multiple eigenvalue having r condition numbers to reflect different sensitivities of the eigenvalues splitted from multiple eigenvalue under a small perturbation. From (3.2) we can introduce the following definition.

Definition 3.5: Let $\|\ \|$ be any unitarily invariant norm and $\lambda_i\left(Y_1^T(\lambda_1^2 E + \lambda_1 F + G)X_1\right)$ be the eigenvalues of $Y_1^T(\lambda_1^2 E + \lambda_1 F + G)X_1$ with

$$\left|\lambda_1\left(Y_1^T(\lambda_1^2 E + \lambda_1 F + G)X_1\right)\right| \ge \cdots \ge \left|\lambda_r\left(Y_1^T(\lambda_1^2 E + \lambda_1 F + G)X_1\right)\right|,$$

Then

$$k_i([M,C,K],\lambda_1) = \sup_{\substack{E,F,G \in C^{n\times n} \\ \|[E,F,G]\|=1}} \left|\lambda_i\left(Y_1^T(\lambda_1^2 E + \lambda_1 F + G)X_1\right)\right|,\ i = 1,\dots,r$$

are called the condition numbers of eigenvalue λ_1.

If we take the Frobenius norm and the spectral norm in Definition 3.5, the corresponding condition numbers are respectively denoted by

$k_i^{(F)}([M,C,K],\lambda_1)$ and $k_i^{(2)}([M,C,K],\lambda_1)$. Obviously,

$k_1([M,C,K],\lambda_1)=\tilde{c}([M,C,K],\lambda_1)$,

$k_1^{(F)}([M,C,K],\lambda_1)=c([M,C,K],\lambda_1)=k_1^{(2)}([M,C,K],\lambda_1)=\tilde{c}^{(2)}([M,C,K],\lambda_1)$.

Next we give an upper bound for condition numbers $k_i([M,C,K],\lambda_1)$.

Theorem 3.3. Let $k_i([M,C,K],\lambda_1)(i=1,...,r)$ be defined in Definition 3.5. Then

$$k_i([M,C,K],\lambda_1)\le\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\sup_{\substack{H\in C^{r\times r}\\ \|H\|\le 1}}\left|\lambda_i(H\Sigma)\right|, i=1,...,r,$$

where $\lambda_i(H\Sigma)(i=1,...,r)$ are the eigenvalues of $H\Sigma$ with $\left|\lambda_1(H\Sigma)\right|\ge\cdots\ge\left|\lambda_r(H\Sigma)\right|$.

Proof. Let $X_1Y_1^T$ have singular value decomposition (3.3), and

$$P=[P_1,P_2],Q=[Q_1,Q_2],P_1,Q_1\in C^{n\times r}, \tag{3.10}$$

Then, by (3.3) we have

$$X_1Y_1^T=P_1\Sigma Q_1^T. \tag{3.11}$$

For any $E,F,G\in C^{n\times n}$ with $\left\|[E,F,G]\right\|=1$, we have

$$\begin{aligned}\left|\lambda_i\left(Y_1^T(\lambda_1^2E+\lambda F+G)X_1\right)\right|&=\left|\lambda_i\left(X_1Y_1^T(\lambda_1^2E+\lambda F+G)\right)\right|\\&=\left|\lambda_i\left(P_1\Sigma Q_1^T(\lambda_1^2E+\lambda F+G)\right)\right|\\&=\left|\lambda_i\left(Q_1^T(\lambda_1^2E+\lambda F+G)P_1\Sigma\right)\right|\\&=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\left|\lambda_i\left(H\Sigma\right)\right|,\end{aligned}$$

where

$$H=\frac{Q_1^H(\lambda_1^2E+\lambda F+G)P_1}{\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}}.$$

From (3.8) and [12] we have

$$\|H\|\le\frac{1}{\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}}\left\|Q_1^H\right\|\left\|P_1\right\|\left\|\lambda_1^2E+\lambda_1F+G\right\|\le 1,$$

Hence,

$$\left\{|\lambda_i(Y_1^T(\lambda_1^2E+\lambda F+G)X_1)|\,\middle|E,F,G\in C^{n\times n},\|[E,F,G]\|=1\right\}$$
$$\subseteq\left\{\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,|\lambda_i(H\Sigma)|\,\middle|H\in C^{r\times r},\|H\|\le 1\right\}.$$

Then the result of this Theorem follows from above inequality.\#

As for condition numbers $k_i^{(2)}([M,C,K],\lambda_1)$ and $k_i^{(F)}([M,C,K],\lambda_1)$, we give their computational expressions as following.

Theorem 3.4. Suppose that $X_1Y_1^T$ has singular value decomposition (3.3) and denote

$$\hat{\sigma}_i=\left(\prod_{j=1}^{i}\sigma_j\right)^{1/i},\ i=1,\dots,r.$$

Then, for $i=1,\dots,r$, we have

$$k_i^{(2)}([M,C,K],\lambda_1)=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\sup_{\substack{H\in C^{r\times r}\\ \|H\|_2\le 1}}|\lambda_i(H\Sigma)|=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,\hat{\sigma}_i, \tag{3.12}$$

$$k_i^{(F)}([M,C,K],\lambda_1)=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\sup_{\substack{H\in C^{r\times r}\\ \|H\|_F\le 1}}|\lambda_i(H\Sigma)|=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,\frac{\hat{\sigma}_i}{\sqrt{i}}. \tag{3.13}$$

Proof. By Theorem 3.3, we have

$$k_i^{(2)}([M,C,K],\lambda_1)\le\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\sup_{\substack{H\in C^{r\times r}\\ \|H\|_2\le 1}}|\lambda_i(H\Sigma)|. \tag{3.14}$$

For any $H\in C^{r\times r}$ with $\|H\|_2\le 1$, we take

$$\hat{E}=\frac{\overline{\lambda_1^2}}{\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}}H,\ \hat{F}=\frac{\overline{\lambda_1}}{\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}}H,\ \hat{G}=\frac{1}{\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}}H,$$
$$E=Q\begin{bmatrix}\hat{E}&O\\O&e_1e_1^T\end{bmatrix}P^H,\ F=Q\begin{bmatrix}\hat{F}&O\\O&O\end{bmatrix}P^H,\ G=Q\begin{bmatrix}\hat{G}&O\\O&O\end{bmatrix}P^H,$$

where e_1 denote the first column of identity matrix of order $n-r$. Then

$\|[E,F,G]\|_2=1$. Let P_1, Q_1 be as in (3.10). Then

$$Q_1^H(\lambda_1^2 E+\lambda_1 F+G)P_1 = Q_1^H Q\begin{bmatrix}\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,H & O\\ O & \lambda_1^2 e_1 e_1^T\end{bmatrix}P^H P_1$$
$$=\begin{bmatrix}I_r & O\end{bmatrix}\begin{bmatrix}\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,H & O\\ O & \lambda_1^2 e_1 e_1^T\end{bmatrix}\begin{bmatrix}I_r\\ O\end{bmatrix}$$
$$=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,H.$$

Further from (3.11), we have

$$\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,|\lambda_i(H\Sigma)| = \left|\lambda_i\left(Q_1^H(\lambda_1^2E+\lambda_1F+G)P_1\Sigma\right)\right|$$
$$=\left|\lambda_i\left(P_1\Sigma Q_1^H(\lambda_1^2E+\lambda_1F+G)\right)\right|$$
$$=\left|\lambda_i\left(X_1Y_1^H(\lambda_1^2E+\lambda_1F+G)\right)\right|$$
$$=\left|\lambda_i\left(Y_1^H(\lambda_1^2E+\lambda_1F+G)X_1\right)\right|.$$

Hence,

$$\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\sup_{\substack{H\in C^{r\times r}\\ \|H\|_2\le 1}}|\lambda_i(H\Sigma)| \le k_i^{(2)}([M,C,K],\lambda_1).$$

Combining above inequality with (3.14) yields that

$$k_i^{(2)}([M,C,K],\lambda_1)=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\sup_{\substack{H\in C^{r\times r}\\ \|H\|_2\le 1}}|\lambda_i(H\Sigma)|. \tag{3.15}$$

According to the proof of Theorem 3.1 in [13], we know that

$$\sup_{\substack{H\in C^{r\times r}\\ \|H\|_2\le 1}}|\lambda_i(H\Sigma)|=\hat{\sigma}_i. \tag{3.16}$$

Thus, (3.12) follows from (3.15) and above equation. Similarly, we have (3.13).\#

Remark 3.1. Note that for any unitarily invariant norm $\|\ \|$, $\|H\|_2\le\|H\|$. Then, from Theorem 3.3 and Theorem 3.4 we see that

$$k_i([M,C,K],\lambda_1)\le k_i^{(2)}([M,C,K],\lambda_1)=\sqrt{|\lambda_1|^4+|\lambda_1|^2+1}\,\hat{\sigma}_i.$$

Now we give an example to validate our results.

Let

$$M=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},\ C=\begin{bmatrix}0 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix},\ K=\begin{bmatrix}-1 & 0 & -3\\ 0 & 0 & 0\\ 0 & 0 & -2\end{bmatrix}.$$

It can be computed that $\lambda_1 = 1$ is a semi-simple eigenvalue $\{M, C, K\}$ with multiplicity 2, and the columns of

$$X_1=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix},\ Y_1=\begin{bmatrix}1/2 & 0\\ 1/2 & 1\\ -3/2 & 0\end{bmatrix},$$

are respectively the right and left eigenvectors corresponding to $\lambda_1 = 1$, and $Y_1^T(2\lambda_1 M + C)X_1 = I_2$.

By Theorem 3.1, Theorem 3.2, Definition 3.4 and Theorem 3.4, we have

$$\begin{aligned}
&c([M,C,K],\lambda_1) = \tilde{c}^{(2)}([M,C,K],\lambda_1) = \hat{c}^{(2)}([M,C,K],\lambda_1) = 2.9408,\\
&\hat{c}^{(\mathrm{F})}([M,C,K],\lambda_1)=3.3541,\\
&k_1^{(2)}\left([M,C,K],\lambda_1\right) = 2.9408,\ k_2^{(2)}\left([M,C,K],\lambda_1\right) = 1.1630,\\
&k_1^{(F)}\left([M,C,K],\lambda_1\right) = 2.9408,\ k_2^{(F)}\left([M,C,K],\lambda_1\right) = 1.1405.
\end{aligned}$$

Now we take E, F, G as the following matrices:

$$E = 10^{-7}\times\begin{bmatrix}-9.193 & 2.513 & -8.227\\ -9.839 & -3.312 & 6.684\\ 9.748 & 6.035 & 2.578\end{bmatrix},$$

$$F = 10^{-7}\times\begin{bmatrix}-9.317 & -5.217 & 1.364\\ -5.522 & 4.842 & -1.291\\ 9.480 & 1.043 & 0.474\end{bmatrix},$$

$$G = 10^{-7}\times\begin{bmatrix}-4.525 & 8.449 & -3.548\\ 7.360 & 6.862 & 3.037\\ 7.947 & 9.338 & 0.462\end{bmatrix}.$$

Assume that $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ are the eigenvalues of $\{M+E, C+F, K+G\}$ splitted from $\lambda_1 = 1$. Straightforward calculations give that

$$\frac{|\tilde{\lambda}_1-\lambda_1|}{\|[E,F,G]\|_2}=2.5027,\quad \frac{|\tilde{\lambda}_2-\lambda_1|}{\|[E,F,G]\|_2}=0.4499,$$
$$\frac{|\tilde{\lambda}_1-\lambda_1|}{\|[E,F,G]\|_F}=1.7789,\quad \frac{|\tilde{\lambda}_2-\lambda_1|}{\|[E,F,G]\|_F}=0.3198.$$

Hence, we get

$$\frac{|\tilde{\lambda}_i-\lambda_1|}{\|[E,F,G]\|_2}\le k_i^{(2)}([M,C,K],\lambda_1)\le \tilde{c}^{(2)}([M,C,K],\lambda_1),\ i=1,2,$$
$$\frac{|\tilde{\lambda}_i-\lambda_1|}{\|[E,F,G]\|_F}\le k_i^{(F)}([M,C,K],\lambda_1)\le c([M,C,K],\lambda_1)\le \hat{c}^{(\mathrm{F})}([M,C,K],\lambda_1),\ i=1,2.$$

Above inequalities show that the condition numbers given in this paper are reasonable.

4 Conclusion

In this paper we give various definitions for condition numbers of semi-simple eigenvalue of regular quadratic eigenvalue problem (Definition 3.2—Definition 3.5) and derive the computational expressions for the introduced condition numbers (Theorem 3.1, Theorem 3.4). The condition numbers defined in this paper can measure not only the worst case sensitivity of semi-simple eigenvalue, but also the different sensitivities of the eigenvalues spawned from semi-simple eigenvalue.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Demmel JW. On condition numbers and the distance to the nearest ill-posed problem. Numer. Math. 1987;51:251-289.

[2] Stewart GW, Zhang G. Eigenvalues of graded matrices and the condition numbers of a multiple eigenvalue. Numerische Mathematik. 1991;58:703-712.

[3] Moro J, Burke JV, Overton ML. On the Lidskii-Vishik-Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure. SIAM J. Matrix Anal. Appl. 1997;18:793-817.

[4] Xie HQ, Dai H. On condition numbers of a nondefective multiple eigenvalue of a nonsymmetric matrix pencil. Linear Algebra and its Applications. 2012;437:1628-1640.

[5] Nichols NK, Kautsky J. Robust eigenstructure assignment in quadratic matrix polynomials: Nonsingular case. SIAM Journal on Matrix Analysis and Applications. 2001;23(1):77-102.

[6] Tisseur F, Meerbergen K. The quadratic eigenvalue Problem. SIAM Review. 2001;43(2):235-286.

[7] Tisseur F. Backward error and condition of polynomial eigenvalue problems. Linear Algebra and its Applications. 2000;309:339-361.

[8] Papathanasiou N, Psarrakos P. On condition numbers of polynomial eigenvalue problems. Applied Mathematics and Computation. 2010;216:1194-1205.

[9] Dedieu JP, Tisseur F. Perturbation theory for homogeneous polynomial eigenvalue problems. Linear Algebra and its Applications. 2003;358:71-94.

[10] Chu KWE. Perturbation of eigenvalues for matrix polynomials via the Bauer-Fike theorems. SIAM Journal on Matrix Analysis and Applications. 2003;25:551-573.

[11] Xie HQ. Sensitivity analysis of semi-simple eigenvalues of regular quadratic eigenvalue problems. Acta Mathematicae Applicatae Sinica, English Series. 2015;31(2):499-518.

[12] Stewart GW, Sun JG. Matrix perturbation theory. New York: Academic Press; 1990.

[13] Sun JG. On worst-case condition numbers of a nondefective multiple eigenvalue. Numerische Mathematik. 1995;69(3):373-382.

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