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Fast Calculation of Linear Finite Element Method for the Stationary Fractional Advection Dispersion Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Author HW carried out the analysis in Section 4 and author ZZ made the main contributions to the other sections. Both authors read and approved the final manuscript.

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Abstract

Appropriate variational formulation and detailed implementation of linear finite element for the stationary fractional advection dispersion equation(FADE) are discussed. Since fractional derivative is nonlocal operator, the stiffness matrix of finite element on traditional variational formulation for FADE is no longer sparse and the computation becomes costly. In this paper, we establish some fractional order integral and differential formulas for linear interpolation basis functions, and then design a special variational formulation which makes the stiffness matrix possess some good properties, such as quasi-symmetry, quasi-sparseness and strictly diagonally domination. These properties are very important in reducing the computational cost and guaranteeing the stability of finite element equations. Numerical examples demonstrating these properties are presented and the applications in contaminant transport in groundwater flow are given.

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1 Introduction

We investigate the implementation of linear finite element to the stationary fractional advection dispersion equation (FADE):

Problem 1: Given $\Omega = (0,1), f : \overline{\Omega} \to \mathbb{R}$, find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$-\mathrm{D}(\lambda\ {}_0\mathrm{D}_x^{-\beta} + (1-\lambda)\ {}_x\mathrm{D}_1^{-\beta})\mathrm{D}u + p(x)\mathrm{D}u + q(x)u \quad = \quad f(x), \quad \text{in } \Omega, \tag{1.1}$$

$$u \quad = \quad 0, \quad \text{on } \partial\Omega, \tag{1.2}$$

where D represents a single spatial derivative; ${}_0\mathrm{D}_x^{-\beta}, {}_x\mathrm{D}_1^{-\beta}$ represent left and right fractional integral operators, respectively, with $0 \le \beta < 1$; $p(x) \in C(\overline{\Omega})$, $q(x) \in C(\overline{\Omega})$, with $q(x) - \frac{1}{2}\mathrm{D}p(x) \ge 0$; $0 \le \lambda \le 1$ (with special interest the values $\lambda = 0, \frac{1}{2}$, and 1).

FADEs have been used in modeling physical phenomena exhibiting anomalous diffusion (see [1], [2], [3], [4]). For example, solutes moving through aquifers do not generally follow a Fickian second-order governing equation, since large deviations from the stochastic process of Brownian motion (see [5], [6], [7]).

Most FADEs have no analytical forms of solutions. Hence the studies on the numerical methods become very important. The numerical solutions of FADEs are rather difficult and remain much challenge although the topics have been studied for many years. Ervin and Roop ([8], [9]) investigated the variational formulation for the FADE and proved the corresponding existence and uniqueness results. Wang et al. established finite difference scheme for fractional diffusion equations by Grünwald-Letnikov derivative([10],[11]). Deng [12] studied finite element methods for solving the space and time fractional Fokker-Plank equation. Li and Xu [13] discussed the weak solution of the space and time fractional diffusion equation. The common technique of these work is to use the following bilinear form

$$B(u,v) := \langle {}_0\mathrm{D}_x^{\alpha}u, {}_x\mathrm{D}_1^{\alpha}v\rangle + \langle p\mathrm{D}u, v\rangle + \langle qu, v\rangle, \tag{1.3}$$

where $\alpha = 1 - \frac{\beta}{2}$, and ${}_0\mathrm{D}_x^{\alpha}$, ${}_x\mathrm{D}_1^{\alpha}$ represent left and right fractional differential operators. The major difference between the fractional derivatives and the classical derivatives is that fractional derivatives are nonlocal operators. So, the stiffness matrix of finite element approximation generated by (1.3) is no longer sparse (see [12], [14]), and the computational cost, storage requirement, and time spend are very expensive. Another difficulty is that the computation stability of finite element equations is not easily proved. Many other research have also studied the numerical scheme to solve fractional differential equations, such as Chen and Pang(see [15],[16],[17]), Khalil and Khan(see [18],[19],[20],[21],[22],[23],[24]), Zhou and Wu [25], Ma and Jiang [26], and references contained therein.

In order to preserve the sparseness of stiffness matrix of classical FEM, our main ideal is to write (1.3) into an equivalent form

$$\begin{aligned} B(u,v) \quad := & \quad \lambda\langle \mathrm{D}u, {}_x\mathrm{D}_1^{-\beta}\mathrm{D}v\rangle + (1-\lambda)\langle \mathrm{D}u, {}_0\mathrm{D}_x^{-\beta}\mathrm{D}v\rangle \\ + & \quad \langle p\mathrm{D}u, v\rangle + \langle qu, v\rangle. \end{aligned} \tag{1.4}$$

Remark 1.1. *It is important to mention that the weak formulation (1.4) is more efficient than weak formulation (1.3) in the sense of computation and analysis. On the one hand, the global dependence* *of the fractional derivative of u is eliminated when generating finite element equations from (1.4), and the stiffness matrix is preserved to be sparse partially . On the other hand, it is more convenient to calculate the fractional derivative of basis functions $v(x) = n_k(x)$ than that of $u(x)$. Because of these advantages, it becomes reality to analyze stiffness matrix, theoretically.*

Benefit from (1.4), we prove that the stiffness matrix possesses some good properties such as symmetry, sparseness, and strictly diagonally domination. To the best of our knowledge, there are no studies in these properties in the literature.

The paper is organized as follows. In Section 2 we recall the definitions and properties of the Riemann-Liouville fractional derivative and fractional integral operators, together with variational formulation over the fractional derivative spaces. In Section 3 we establish some formulas for the linear interpolation basis functions. In Section 4, we investigate the structure of stiffness matrix generated by Galerkin approximation. In Section 5, numerical results are presented to verify the convergence and fast computation. Concluding remarks are given in the final section.

2 Fractional Derivative Space and Variational Formulation

In this section, we first give some definitions and basic properties which will be used to construct the finite element equations in Section 3 and Section 4. For more results, we refer the reader to Podlubny [27] or other books on basic fractional calculus (see e.g. [28], [29]).

Definition 2.1. *(***Riemann-Liouville Fractional Integral).** *Let u be a function defined on $\Lambda = (a, b)$ and $\sigma > 0$. Then the left and right Riemann-Liouville fractional integral of order σ are defined to be*

$$ {}_a\mathrm{D}_x^{-\sigma} u(x) := \frac{1}{\Gamma(\sigma)} \int_a^x (x-s)^{\sigma-1} u(s) ds. \tag{2.1} $$

$$ {}_x\mathrm{D}_b^{-\sigma} u(x) := \frac{1}{\Gamma(\sigma)} \int_x^b (s-x)^{\sigma-1} u(s) ds. \tag{2.2} $$

Definition 2.2. *(***Riemann-Liouville Fractional Derivative).** *Let u be a function defined on $\Lambda = (a, b)$ and $\mu > 0$, n be the smallest integer than $\mu (n-1 \le \mu < n)$, and $\sigma = n - \mu$. Then the left and right Riemann-Liouville fractional derivative of order μ are defined to be*

$$ {}_a\mathrm{D}_x^{\mu} u(x) := \frac{1}{\Gamma(\sigma)} \frac{d^n}{dx^n} \int_a^x (x-s)^{\sigma-1} u(s) ds, \tag{2.3} $$

$$ {}_x\mathrm{D}_b^{\mu} u(x) := \frac{(-1)^n}{\Gamma(\sigma)} \frac{d^n}{dx^n} \int_x^b (s-x)^{\sigma-1} u(s) ds. \tag{2.4} $$

Proposition 2.3. *The left and right Riemann-Liouville fractional integral and differential operators satisfy the following properties.*

(i) **(Semigroup Property)**. *The left and right Riemann-Liouville fractional integral operators follow the semigroup properties, i.e.,*

$$ {}_a\mathrm{D}_x^{-\mu} {}_a\mathrm{D}_x^{-\sigma} u(x) := {}_a\mathrm{D}_x^{-\mu-\sigma} u(x), \ \forall x \in \Lambda, \quad \forall \mu, \sigma > 0, \tag{2.5} $$

$$ {}_x\mathrm{D}_b^{-\mu} {}_x\mathrm{D}_b^{-\sigma} u(x) := {}_x\mathrm{D}_b^{-\mu-\sigma} u(x), \ \forall x \in \Lambda, \quad \forall \mu, \sigma > 0. \tag{2.6} $$

(ii) **(Adjoint Property)**. *The left and right Riemann-Liouville fractional integral operators are adjoints in the L^2 sense, i.e., for all $\sigma > 0$*

$$ ({}_a\mathrm{D}_x^{-\sigma} u, v) = (u, {}_x\mathrm{D}_b^{-\sigma} v), \quad \forall u, v \in L^2(\Lambda). \tag{2.7} $$

(iii) **(Composition Property)**. *Let $\mu > 0$, $u \in C^{\infty}(\Lambda)$, $\Lambda \subset \mathbb{R}$. The following composition rules hold for the left and right Riemann-Liouville fractional integral and differential operators:*

$$ {}_a\mathrm{D}_x^{\mu}\, {}_a\mathrm{D}_x^{-\mu} u(x) := u(x), \tag{2.8}$$

$$ {}_x\mathrm{D}_b^{\mu}\, {}_x\mathrm{D}_b^{-\mu} u(x) := u(x), \tag{2.9}$$

$$ {}_a\mathrm{D}_x^{-\mu}\, {}_a\mathrm{D}_x^{\mu} u(x) := u(x), \quad \forall u(x) \text{ such that } \overline{supp(u)} \subset \Lambda, \tag{2.10}$$

$$ {}_x\mathrm{D}_b^{-\mu}\, {}_x\mathrm{D}_b^{\mu} u(x) := u(x), \quad \forall u(x) \text{ such that } \overline{supp(u)} \subset \Lambda. \tag{2.11}$$

Now we turn to the variational formulation for Problem 1. For the analysis of the approximation to FADE, we introduce associated left and right fractional derivative spaces $J^{\mu}_{L,0}(\Lambda), J^{\mu}_{R,0}(\Lambda)$, and point out the equivalence of these spaces with fractional order Hilbert space $H^{\mu}_0(\Lambda)$.

Definition 2.4. (Fractional Derivative Space). *Let $\mu > 0$. Define the semi-norm and norm*

$$ |u|_{J^{\mu}_{L,0}} := \|{}_a\mathrm{D}_x^{\mu} u(\cdot)\|_{L^2(\Lambda)}, \quad \|u\|_{J^{\mu}_{L,0}} := \left(\|u\|^2_{L^2(\Lambda)} + |u|^2_{J^{\mu}_{L,0}}\right)^{1/2}, \tag{2.12}$$

and let $J^{\mu}_{L,0}(\Lambda)$ denote the closure of $C_0^{\infty}(\Lambda)$ with respect to $\|\cdot\|_{J^{\mu}_{L,0}}$. The definition of $J^{\mu}_{R,0}(\Lambda)$ is similar.

Definition 2.5. (Fractional Order Hilbert Space). *Let $\mu > 0$. Define the semi-norm and norm*

$$ |u|_{H^{\mu}_0(\Lambda)} := \||\omega|^{\mu}\hat{u}(\omega)\|_{L^2(\Lambda)}, \quad \|u\|_{H^{\mu}_0(\Lambda)} := \left(\|u\|^2_{L^2(\Lambda)} + |u|^2_{H^{\mu}_0(\Lambda)}\right)^{1/2}, \tag{2.13}$$

where $\hat{u}(\omega)$ is the Fourier transform of function u, and let $H^{\mu}_0(\Lambda)$ denote the closure of $C_0^{\infty}(\Lambda)$ with respect to $\|\cdot\|_{H^{\mu}_0(\Lambda)}$.

Ervin and Roop [8] proved the following equivalence theorem.

Lemma 2.6. *Let $\mu > 0$. Then the spaces $J^{\mu}_{L,0}(\Lambda)$, $J^{\mu}_{R,0}(\Lambda)$ and $H^{\mu}_0(\Lambda)$ are equal. And, if $\mu \neq n - \frac{1}{2}$, $n \in \mathbb{N}$, the spaces $J^{\mu}_{L,0}(\Lambda)$, $J^{\mu}_{R,0}(\Lambda)$ and $H^{\mu}_0(\Lambda)$ have equivalent semi-norms and norms.*

Li and Xu [13] gave the adjoint property of fractional derivative:

Lemma 2.7. *For all positive real μ, if $u \in H^{\mu}_0, v \in C_0^{\infty}(\Lambda)$, then*

$$ \left({}_0\mathrm{D}_x^{\mu} u(x), v(x)\right) = \left(u(x), {}_x\mathrm{D}_1^{\mu} v(x)\right)\right). $$

Let $\Omega = (0,1)$ and $0 \le \beta < 1$. Define $\alpha := 1 - \beta/2$, so that $1/2 < \alpha \le 1$. Over $H^{\alpha}_0(\Omega)$ space, by integrating by parts, we well define the associated bilinear form $B : (H^{\alpha}_0(\Omega) \cap H^1_0(\Omega)) \times (H^{\alpha}_0(\Omega) \cap H^1_0(\Omega)) \to \mathbb{R}$ as

$$ \begin{aligned} B(u,v) \quad := & \quad \lambda\langle {}_0\mathrm{D}_x^{-\beta}\mathrm{D}u, \mathrm{D}v\rangle + (1-\lambda)\langle {}_x\mathrm{D}_1^{-\beta}\mathrm{D}u, \mathrm{D}v\rangle \\ + & \quad \langle p(x)\mathrm{D}u, v\rangle + \langle q(x)u, v\rangle, \end{aligned} \tag{2.14}$$

where $(\cdot)$ denotes the product on $L^2(\Omega)$, and $\langle\cdot\rangle$ the duality of $H^{-\alpha}(\Omega)$ and $H^{\alpha}(\Omega)$.

Applying semi-group, adjoint, composition properties of fractional integral operators(see Proposition 2.3) and Lemma 2.7, we may write the bilinear form $B: H^{\alpha}_0(\Omega) \times H^{\alpha}_0(\Omega) \to \mathbb{R}$ as

$$ \begin{aligned} B(u,v) \quad := & \quad \lambda\langle {}_0\mathrm{D}_x^{\alpha} u, \ {}_x\mathrm{D}_1^{\alpha} v\rangle + (1-\lambda)\langle {}_x\mathrm{D}_1^{\alpha} u, \ {}_0\mathrm{D}_x^{\alpha} v\rangle \\ + & \quad \langle p(x) {}_x\mathrm{D}_1^{\frac{1}{2}} u, {}_x\mathrm{D}_1^{\frac{1}{2}} v\rangle + \langle q(x)u, v\rangle. \end{aligned} \tag{2.15}$$

For a given $f \in H^{-\alpha}(\Omega)$, we define the associated linear functional $F: H^{\alpha}_0(\Omega) \to \mathbb{R}$ as

$$ F(v) \quad := \quad \langle f, v\rangle. \tag{2.16}$$

Thus, the Galerkin variational solution of Problem 1 can be defined as follows.

Definition 2.8. (Variational Solution). *A function $u \in H_0^\alpha(\Omega)$ is a variational solution of Problem 1 provided that*

$$B(u,v) = F(v), \quad \forall v \in H_0^\alpha(\Omega). \tag{2.17}$$

Remark 2.9. *Based on (2.15), Evin and Roop [8] have proved the coercivity of $B(\cdot,\cdot)$ and the continuity of $B(\cdot,\cdot)$ and $F(\cdot)$ over space $H_0^\alpha(\Omega)$. So, applying the Lax-Milgram theorem (see e.g., [30]), there exists a unique solution $u \in H_0^\alpha(\Omega)$ to the variational problem (2.17).*

However, for the computation convenience, we suggest to write (2.15) into another equivalent form $B:\ (H_0^\alpha(\Omega)\cap H_0^1(\Omega)) \times (H_0^\alpha(\Omega)\cap H_0^1(\Omega)) \to \mathbb{R}$ as

$$\begin{aligned} B(u,v) \quad &:= \quad \lambda\langle \mathrm{D}u, {}_x\mathrm{D}_1^{-\beta}\mathrm{D}v\rangle + (1-\lambda)\langle \mathrm{D}u, {}_0\mathrm{D}_x^{-\beta}\mathrm{D}v\rangle \\ &+ \quad \langle p(x)\mathrm{D}u, v\rangle + \langle q(x)u, v\rangle. \end{aligned} \tag{2.18}$$

All advantages of using bilinear form (2.18) are explained in Remark 1.1. Benefit from (2.18), we prove that the stiffness matrix possesses some good properties such as symmetry, sparseness, and strictly diagonally domination in the following sections.

3 Some Formulas of Linear Interpolation Basis Functions

In this section, we establish some fractional integral and differential formulas for linear interpolation basis functions, which are crucial to the analysis of stiffness matrix in Section 4.

Lemma 3.1. *For $\beta > 0$, the following right and left Riemann-Liouville fractional integral formulations for power functions x^n hold*

(i) ${}_x\mathrm{D}_b^{-\beta}x^0 = \frac{(b-x)^\beta}{\Gamma(\beta+1)}$;

(ii) ${}_x\mathrm{D}_b^{-\beta}x = \frac{b(b-x)^\beta}{\Gamma(\beta+1)} - \frac{(b-x)^{\beta+1}}{\Gamma(\beta+2)}$;

(iii) ${}_x\mathrm{D}_b^{-\beta}x^2 = \frac{b^2(b-x)^\beta}{\Gamma(\beta+1)} - \frac{2b(b-x)^{\beta+1}}{\Gamma(\beta+2)} + \frac{2(b-x)^{\beta+2}}{\Gamma(\beta+3)}$;

(iv) ${}_x\mathrm{D}_b^{-\beta}x^n = \frac{b^n(b-x)^\beta}{\Gamma(\beta+1)} - n{}_x\mathrm{D}_b^{-(\beta+1)}x^{n-1}, n = 1,2,\cdots$;

(v) ${}_a\mathrm{D}_x^{-p}(x-a)^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+p+1)}(x-a)^{\mu+p},\ p>0, \mu>-1.$

Proof. The results (i)-(iv) follow directly from the definition of right Riemann-Liouville fractional integral, and (v) is proved in [27].

For the simpleness, we denote the uniform spatial mesh

$$\mathcal{M}:\ 0 = x_0 < x_1 \cdots < x_N = 1$$

with equal step length $h = 1/N$. And define $n_k(x), (k = 1, 2, \cdots, N-1)$ the basis functions for the class of continuous piecewise linear functions on the interval Ω,

$$n_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k], \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}], \\ 0, & \text{others}. \end{cases}$$

Lemma 3.2. *Let $\beta > 0$, $n_k(x)$ are the basis functions and $x \in [x_{i-1}, x_i]$, then*

$${}_0\mathrm{D}_x^{-\beta}\mathrm{D}n_k(x) = \frac{1}{h\Gamma(1+\beta)} \times$$

$$\begin{cases} (x-x_{k-1})^\beta + (x-x_{k+1})^\beta - 2(x-x_k)^\beta, & 1 \le k \le i-2, \\ (x-x_{i-2})^\beta - 2(x-x_{i-1})^\beta, & k = i-1, \\ (x-x_{i-1})^\beta, & k = i, \\ 0, & i+1 \le k \le N-1. \end{cases} \tag{3.1}$$

Proof. In the following proof, we use the left fractional derivatives formulas of power functions in lemma 3.1.

(i) Case $i+1 \le k \le N-1$. Since $\mathrm{D}n_k(x) \equiv 0$ on the interval $[0, x_i]$, we have

$$ {}_0\mathrm{D}_x^{-\beta}\mathrm{D}n_k(x) = 0. $$

(ii) Case $k = i$. For all $x \in [x_{i-1}, x_i]$, we have

$$ {}_0\mathrm{D}_x^{-\beta}\mathrm{D}n_k(x) = \frac{1}{h}\,{}_{x_{i-1}}\mathrm{D}_x^{-\beta}x^0 = \frac{1}{h\Gamma(\beta+1)}(x-x_{i-1})^\beta. $$

(iii) Case $k = i-1$. For all $x \in [x_{i-1}, x_i]$, we have

$$ \begin{aligned} {}_0\mathrm{D}_x^{-\beta}\mathrm{D}n_k(x) &= \frac{1}{h}\left[{}_{x_{i-2}}\mathrm{D}_x^{-\beta}x^0 - {}_{x_{i-1}}\mathrm{D}_x^{-\beta}x^0\right] - \frac{1}{h}\,{}_{x_{i-1}}\mathrm{D}_x^{-\beta}x^0 \\ &= \frac{1}{h\Gamma(\beta+1)}\left[(x-x_{i-2})^\beta - 2(x-x_{i-1})^\beta\right]. \end{aligned} $$

(iv) Case $1 \le k \le i-2$. For all $x \in [x_{i-1}, x_i]$, we have

$$ \begin{aligned} {}_0\mathrm{D}_x^{-\beta}\mathrm{D}n_k(x) &= \frac{1}{h}\left[{}_{x_{k-1}}\mathrm{D}_x^{-\beta}x^0 - {}_{x_k}\mathrm{D}_x^{-\beta}x^0\right] - \frac{1}{h}\left[{}_{x_k}\mathrm{D}_x^{-\beta}x^0 - {}_{x_{k+1}}\mathrm{D}_x^{-\beta}x^0\right] \\ &= \frac{1}{h\Gamma(\beta+1)}\left[(x-x_{k-1})^\beta + (x-x_{k+1})^\beta - 2(x-x_k)^\beta\right]. \end{aligned} $$

The proof of lemma 3.2 is completed.

Lemma 3.3. *Let $\beta > 0$, $n_k(x)$ are the basis functions and $x \in [x_{i-1}, x_i]$. Then*

$$ {}_x\mathrm{D}_1^{-\beta}\mathrm{D}n_k(x) = \frac{1}{h\Gamma(1+\beta)}\times $$

$$ \begin{cases} 2(x_k-x)^\beta - (x_{k+1}-x)^\beta - (x_{k-1}-x)^\beta, & i+1 \le k \le N-1, \\ 2(x_i-x)^\beta - (x_{i+1}-x)^\beta, & k = i, \\ -(x_i-x)^\beta, & k = i-1, \\ 0, & 1 \le k \le i-2. \end{cases} \tag{3.2} $$

Proof. The proof is very similar to the proof of Lemma 3.2. The only difference is that we use the right fractional derivatives formulas of power functions in Lemma 3.1.

For convenience, we denote the integral average of ${}_0\mathrm{D}_x^{-\beta}\mathrm{D}n_k(x)$ and ${}_x\mathrm{D}_1^{-\beta}\mathrm{D}n_k(x)$ as follows:

$$ I_{k,i}^l(\beta) \quad := \quad \frac{1}{h}\int_{x_{i-1}}^{x_i} {}_0\mathrm{D}_x^{-\beta}\mathrm{D}n_k(x)dx, \tag{3.3} $$

and

$$ I_{k,i}^r(\beta) \quad := \quad \frac{1}{h}\int_{x_{i-1}}^{x_i} {}_x\mathrm{D}_1^{-\beta}\mathrm{D}n_k(x)dx. \tag{3.4} $$

Note that symbols l and r stand for left Riemann-Liouville integral operator and right Riemann-Liouville integral operator, respectively.

Applying the results of Lemma 3.2 and Lemma 3.3, and by elemental calculations we obtain the following lemmas which are very important in the next analysis.

Lemma 3.4. *Let $\beta > 0$ and $n_k(x)$ be the basis functions. Then*

$$I^l_{k,i}(\beta) \quad = \quad \frac{h^{\beta-1}}{\Gamma(2+\beta)} \times \begin{cases} S^l_{k,i} & if \;\; 1 \le k \le i-2, \\ 2^{\beta+1}-3 & if \;\; k = i-1, \\ 1 & if \;\; k = i, \\ 0 & if \;\; i+1 \le k \le N-1, \end{cases} \tag{3.5}$$

where $S^l_{k,i}(\beta) = (i-k+1)^{\beta+1} - 3(i-k)^{\beta+1} + 3(i-k-1)^{\beta+1} - (i-k-2)^{\beta+1}$.

Lemma 3.5. *Let $\beta > 0$ and $n_k(x)$ be the basis functions. Then*

$$I^r_{k,i}(\beta) \quad = \quad \frac{-h^{\beta-1}}{\Gamma(2+\beta)} \times \begin{cases} S^r_{k,i} & if \;\; i+1 \le k \le N-1, \\ 2^{\beta+1}-3 & if \;\; k = i, \\ 1 & if \;\; k = i-1, \\ 0 & if \;\; 1 \le k \le i-2, \end{cases} \tag{3.6}$$

where $S^r_{k,i}(\beta) = -(k-i-1)^{\beta+1} + 3(k-i)^{\beta+1} - 3(k-i+1)^{\beta+1} + (k-i+2)^{\beta+1}$.

Now we give some useful equalities between $I^l_{k,i}(\beta)$ and $I^r_{k,i}(\beta)$ or between $S^l_{k,i}(\beta)$ and $S^r_{k,i}(\beta)$.

Theorem 3.6. *Let $\beta > 0$ and $n_k(x)$ be the basis functions. Then we have*

(i) For $1 \le i \le N-1$, and $1 \le k \le N-2$, we have

$$I^l_{k,i}(\beta) = I^l_{k+1,i+1}(\beta). \tag{3.7}$$

(ii) For $1 \le i \le N-1$, and $1 \le k \le N-2$, we have

$$I^r_{k,i}(\beta) = I^r_{k+1,i+1}(\beta). \tag{3.8}$$

(iii) For $2 \le i \le N-1$, we have

$$S^r_{i,i-1}(\beta) = S^l_{i-1,i+1}(\beta). \tag{3.9}$$

(iv) For $1 \le i \le N-1$, and $1 \le k \le i-2$, we have

$$S^l_{k,i}(\beta) - S^l_{k,i+1}(\beta) = S^r_{i,k}(\beta) - S^r_{i,k+1}(\beta). \tag{3.10}$$

(v) For $1 \le i \le N-1$, and $1 \le k \le N-1$, we have

$$I^l_{k,i}(\beta) - I^l_{k,i+1}(\beta) = I^r_{i,k}(\beta) - I^r_{i,k+1}(\beta). \tag{3.11}$$

Proof. The results follow directly from (3.5) and (3.6).

4 Structures of the Stiffness Matrix

In this section, we first introduce the existence, uniqueness and error estimates of Galerkin approximation, and then discuss the structure of stiffness matrix. The analysis shows that stiffness matrix is sparse, symmetrical and strictly diagonally dominant under some conditions.

Associated with the uniform mesh $\mathcal{M} : 0 = x_0 < x_1 < \cdots < x_N = 1$, define the finite-dimensional subspace $X_h \subset H^\alpha_0(\Omega)$ as

$$X_h := \{v \in H^\alpha_0(\Omega) \cap C^0(\overline{\Omega}) | v = \sum_{k=1}^{N-1} v_k n_k(x), \forall v_k \in \mathbb{R}\}. \tag{4.1}$$

Let $u_h = \sum_{i=1}^{N-1} u_i n_i(x)$, where u_i are the expected nodal values of approximate solution, be the solution to the finite-dimensional variational problem:

$$B(u_h, v_h) = F(v_h), \forall v_h \in X_h. \tag{4.2}$$

Note that the existence and uniqueness of solutions to (4.2) follow from the fact that X_h is a subset of the space $H_0^\alpha(\Omega) \cap H_0^1(\Omega)$ (see [30]). Under the assumptions on the regularity of the solutions to the adjoint problem of Problem 1, Ervin and Roop [8] have obtained the convergence estimate in L^2 norm.

Theorem 4.1. *Let $u \in H_0^\alpha(\Omega) \bigcap H^r(\Omega)(\alpha \le r \le m)$ solve (2.17) and u_h solve (4.2), where $m-1$ is the degree of Galerkin finite element approximation. Then, if the regularity of the solution to the adjoint problem is satisfied, then there exits a constant C such that the error $e = u - u_h$ satisfies*

$$\|e\|_{L^2(\Omega)} \le Ch^r \|u\|_{H_0^r(\Omega)}, \quad \alpha \ne 3/4, \tag{4.3}$$

$$\|e\|_{L^2(\Omega)} \le Ch^r \|u\|_{H_0^{r-\varepsilon}(\Omega)}, \quad \alpha = 3/4, 0 < \forall \varepsilon < 1/2. \tag{4.4}$$

In the case of linear finite element $m = 2$, the convergence rate is 2. In Section 5, we will give numerical experiments to support the result.

Now, we turn to discuss the structure of stiffness matrix. For the Galerkin method, the test functions v_h are chosen successively to be each of the nodal basis functions n_k, $k = 1, 2, \cdots, N-1$ such that

$$B(u_h, n_k) = F(n_k), k = 1, 2, \cdots, N-1. \tag{4.5}$$

Equations (4.5) can be written in the matrix form (Stiffness Matrix and Load Vector form) as

$$\mathbf{Au} = \mathbf{F}, \tag{4.6}$$

where

$$\mathbf{A} = [a_{ij}]_{N-1,N-1}, \ \mathbf{u} = [u_1, u_2, \cdots, u_{N-1}]^T, \ \mathbf{F} = [f_1, f_2, \cdots, f_{N-1}]^T,$$

and

$$f_k = \langle f, n_k \rangle_\Omega.$$

From (2.18) the bilinear form $B(u_h, n_k)$ can be divided into three parts,

$$B_1(u_h, n_k) := \lambda \langle \mathrm{D}u_h, {}_x\mathrm{D}_1^{-\beta} \mathrm{D}n_k \rangle,$$
$$B_2(u_h, n_k) := (1-\lambda) \langle \mathrm{D}u_h, {}_0\mathrm{D}_x^{-\beta} \mathrm{D}n_k \rangle,$$

and

$$B_3(u_h, n_k) := \langle p(x)\mathrm{D}u_h, n_k \rangle + \langle q(x) u_h, n_k \rangle.$$

Thus, the stiffness matrix $\mathbf{A}$ can also be divided into three parts,

$$\begin{aligned} \mathbf{A} &= \lambda \mathbf{A}_1 + (1-\lambda)\mathbf{A}_2 + \mathbf{A}_3 \\ &= \lambda [a_{ki}^{(1)}]_{N-1,N-1} + (1-\lambda)[a_{ki}^{(2)}]_{N-1,N-1} + [a_{ki}^{(3)}]_{N-1,N-1}, \end{aligned}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are generated by $B_1(\cdot,\cdot)$, $B_2(\cdot,\cdot)$ and $B_3(\cdot,\cdot)$, respectively. It is well-known that $\mathbf{A}_3$ generated by $B_3(\cdot,\cdot)$ is a tridiagonal matrix. So, we only need to discuss the form of $\mathbf{A}_1$ and $\mathbf{A}_2$.

In following theorems we give the sparseness and symmetry of the matrix $\mathbf{A}$.

Theorem 4.2. *The elements of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ have following expressions*

$$a_{ki}^{(1)} = I_{k,i}^r(\beta) - I_{k,i+1}^r(\beta), \tag{4.7}$$

$$a_{ki}^{(2)} = I_{k,i}^l(\beta) - I_{k,i+1}^l(\beta). \tag{4.8}$$

Proof. From the definition of $u_h(x)$, Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned}
\langle \mathrm{D}u_h, {}_x\mathrm{D}_1^{-\beta}\mathrm{D}n_k\rangle_{e_i} &= \int_{x_{i-1}}^{x_i} (-u_{i-1}\frac{1}{h} + u_i\frac{1}{h})_x\mathrm{D}_1^{-\beta} Dn_k(x)dx \\
&= (-u_{i-1} + u_i)\frac{1}{h}\int_{x_{i-1}}^{x_i} {}_x\mathrm{D}_1^{-\beta}\mathrm{D}n_k(x)dx \\
&= (-u_{i-1} + u_i)I^r_{k,i}(\beta), \qquad (4.9)\\
\langle \mathrm{D}u_h, {}_x\mathrm{D}_1^{-\beta}\mathrm{D}n_k\rangle_{e_{i+1}} &= \int_{x_i}^{x_{i+1}} (-u_i\frac{1}{h} + u_{i+1}\frac{1}{h})_x\mathrm{D}_1^{-\beta}\mathrm{D}n_k(x)dx \\
&= (-u_i + u_{i+1})\frac{1}{h}\int_{x_i}^{x_{i+1}} {}_x\mathrm{D}_1^{-\beta}\mathrm{D}n_k(x)dx \\
&= (-u_i + u_{i+1})I^r_{k,i+1}(\beta). \qquad (4.10)
\end{aligned}$$

The terms in bilinear form $B_2(u_h, n_k)$ is analogous to the above expression:

$$\langle \mathrm{D}u_h, {}_0\mathrm{D}_x^{-\beta}\mathrm{D}n_k\rangle_{e_i} = (-u_{i-1} + u_i)I^l_{k,i}(\beta), \qquad (4.11)$$

$$\langle \mathrm{D}u_h, {}_0\mathrm{D}_x^{-\beta}\mathrm{D}n_k\rangle_{e_{i+1}} = (-u_i + u_{i+1})I^l_{k,i+1}(\beta). \qquad (4.12)$$

Thus, from (4.9)-(4.12) we have

$$a^{(1)}_{ki} = I^r_{k,i}(\beta) - I^r_{k,i+1}(\beta), \quad a^{(2)}_{ki} = I^l_{k,i}(\beta) - I^l_{k,i+1}(\beta),$$

which end the proof.

Using (3.7) and (3.8) repeatedly the following corollary holds:

Corollary 4.3. *All elements $a^{(1)}_{ik}$ and $a^{(2)}_{ik}$ on the same diagonal line are equal, i.e.,*

$$a^{(1)}_{ki} = a^{(1)}_{k+j,i+j}, \forall \ \ 1 \le i, i+j \le N-1, 1 \le k, k+j \le N-1, \qquad (4.13)$$

$$a^{(2)}_{ki} = a^{(2)}_{k+j,i+j}, \forall \ \ 1 \le i, i+j \le N-1, 1 \le k, k+j \le N-1. \qquad (4.14)$$

Theorem 4.4. *The stiffness matrixes $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}$ have the following properties:*

(i) $\boldsymbol{A}_1$ is a lower Hessenberg matrix;
(ii) $\boldsymbol{A}_2$ is an upper Hessenberg matrix;
(iii) If $\lambda = 1$, then $\boldsymbol{A}$ is a lower Hessenberg matrix;
(iv) If $\lambda = 0$, then $\boldsymbol{A}$ is an upper Hessenberg matrix.

Proof. Results (i) and (ii) follow directly from (3.5), (3.6), (4.7) and (4.8), and (iii), (iv) follow the fact that $\mathbf{A}_3$ is a tridiagonal matrix.

Theorem 4.5. (Quasi-symmetrical Property) *The stiffness matrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}$ have the following properties*

(i) $\boldsymbol{A}_1 = \boldsymbol{A}_2^T$ and $\boldsymbol{A}_1 + \boldsymbol{A}_2$ is symmetric;
(ii) If $\lambda = \frac{1}{2}$, $p(x) = 0$, and $q(x)$ is a constant, then $\boldsymbol{A}$ is a symmetric matrix.

Proof. (i) Applying the property (3.11), (4.7) and (4.8), we get

$$\begin{aligned}
\mathbf{A}_1 &= [I^r_{k,i}(\beta) - I^r_{k,i+1}(\beta)]_{N-1,N-1} \\
&= [I^l_{i,k}(\beta) - I^l_{i,k+1}(\beta)]_{N-1,N-1} \\
&= \mathbf{A}_2^T.
\end{aligned}$$

(ii) Since $p(x) = 0$ and $q(x)$ is a constant, $\mathbf{A}_3$ is a symmetric matrix. So, from (i) we have

$$\mathbf{A}^T = (\frac{1}{2}\mathbf{A}_1 + \frac{1}{2}\mathbf{A}_2 + \mathbf{A}_3)^T = \frac{1}{2}\mathbf{A}_2 + \frac{1}{2}\mathbf{A}_1 + \mathbf{A}_3 = \mathbf{A},$$

which means $\mathbf{A}$ is a symmetric matrix. The proof is completed.

Combining Corollary 4.3, Theorem 4.4 and Theorem 4.5, the stiffness matrix $\mathbf{A}$ has the following structure

$$\mathbf{A} = \lambda \mathbf{H}^T + (1-\lambda)\mathbf{H} + \mathbf{D},$$

where $\mathbf{D} = \mathbf{A}_3$ is a tridiagonal matrix and $\mathbf{H}^T$ is an upper Hessenberg matrix with the form

$$\mathbf{H} \quad = \quad \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{N-3} & c_{N-2} & c_{N-1} \\ c_{-1} & c_1 & c_2 & c_3 & \cdots & c_{N-3} & c_{N-2} \\ 0 & c_{-1} & c_1 & c_2 & c_3 & \cdots & c_{N-3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{-1} & c_1 & c_2 & c_3 \\ 0 & 0 & \cdots & 0 & c_{-1} & c_1 & c_2 \\ 0 & 0 & 0 & \cdots & 0 & c_{-1} & c_1 \end{bmatrix}. \tag{4.15}$$

Remark 4.6. (Quasi-sparse Property) *(i) In the process of developing stiffness matrix $\boldsymbol{A}$, we only need to compute and save the elements c_{-1}, c_1, $c_2, \cdots, c_{N-1}$ and a tridiagonal matrix $\boldsymbol{D}$, which greatly reduce the computation cost and storage requirement.*

(ii) Expressions (4.7) and (4.8) mean that we can compute $c_i\,(i = -1, 1, \cdots, N-1)$ directly, i.e.,

$$c_{-1} = I^l_{2,1}(\beta) - I^l_{2,2}(\beta), \quad c_i = I^l_{1,i}(\beta) - I^l_{1,i+1}(\beta), \quad i = 1, 2, \cdots, N-1. \tag{4.16}$$

Finally, we discuss the strictly diagonally dominant property of $\mathbf{A}$, which ensures the stability of the finite element equations (4.6).

Theorem 4.7. *Let $0 < \beta < 1$. Then the matrix $\boldsymbol{H}$ generated by $B_1(u_h, v_k)$ and $B_2(u_h, v_k)$ has the following properties:*

(i) For the off-diagonal elements of $\boldsymbol{H}$, $c_i < 0, (i = -1, 2, 3, \cdots N-1)$; and for the diagonal elements, $c_1 > 0$.
(ii) $\boldsymbol{H}$ is a strictly diagonally dominant matrix.

Proof. (i) From (4.16), we have

$$c_i = I^l_{1,i}(\beta) - I^l_{1,i+1}(\beta) = \frac{h^{\beta-1}}{\Gamma(\beta+2)} \times \begin{cases} S^l_{1,i} - S^l_{1,i+1} & \text{if} \quad i \geq 3, \\ 2^{\beta+1} - 3 - S^l_{1,i+1} & \text{if} \quad i = 2, \\ 4 - 2^{\beta+1} & \text{if} \quad i = 1, \end{cases}$$

$$c_{-1} = I^l_{2,1}(\beta) - I^l_{2,2}(\beta) = \frac{-h^{\beta-1}}{\Gamma(\beta+2)}.$$

It is not difficult to verify that $c_{-1} < 0, c_i < 0 (i = 2, 3, \cdots, N-1)$ and $c_1 > 0$.

(ii) We only need to prove that

$$\Delta := |c_1| - \left(|c_{-1}| + \sum_{i=2}^{N-1} |c_i|\right) > 0.$$

Indeed,

$$\begin{aligned} |c_{-1}| + \sum_{i=2}^{N-1} |c_i| &= \frac{h^{\beta-1}}{\Gamma(\beta+2)} \left(\sum_{i=3}^{N-1} (S^l_{1,i+1} - S^l_{1,i}) - (2^{\beta+1} - 3 - S^l_{1,3}) + 1 \right) \\ &= \frac{h^{\beta-1}}{\Gamma(\beta+2)} \left(S^l_{1,N} + 4 - 2^{\beta+1} \right). \end{aligned}$$

So

$$\begin{aligned} \Delta &= -\frac{h^{\beta-1}}{\Gamma(\beta+2)} S^l_{1,N} \\ &= \frac{h^{\beta-1}}{\Gamma(\beta+2)} \left[(N-3)^{\beta+1} - 3(N-2)^{\beta+1} + 3(N-1)^{\beta+1} - N^{\beta+1} \right] \\ &= \frac{h^{\beta-1}}{\Gamma(\beta+2)} \left(G(N-1) - G(N) \right), \end{aligned} \tag{4.17}$$

where

$$G(x) = x^{\beta+1} - 2(x-1)^{\beta+1} + (x-2)^{\beta+1}.$$

Due to the convexity of function $y(x) = x^\beta$, ($x > 0$ and $0 < \beta < 1$), we have

$$G'(x) = \frac{2}{\beta+1} \left[\frac{x^\beta + (x-2)^\beta}{2} - (x-1)^\beta \right] < 0, \quad \forall x \geq 2,$$

which means that $G(x)$ is a strictly monotonically decreasing function on $[2, +\infty]$. So we have $G(N-1) - G(N) > 0$, and from (4.17) we obtain that

$$\Delta = |\tilde{\Lambda}_1| - \left(|c_{-1}| + \sum_{i=2}^{N-1} |c_i| \right) > 0,$$

which ends the proof.

Theorem 4.8. (Strictly Diagonally Dominated Property). *Let $0 < \beta < 1$, $0 \leq \lambda \leq 1$, $q(x) = constant \geq 0$, $p(x)$ be a constant, and $|p| \leq \Delta$, where Δ is defined by (4.17). Then the stiffness matrix $\boldsymbol{A}$ is strictly diagonally dominant. Hence, the numerical scheme for (4.5) is stable.*

Proof. If $q(x) = 0$, then $\mathbf{A}_3 = \mathbf{D}_1$ has the form as

$$\mathbf{D}_1 = \frac{p}{2} \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Also, from Theorem 4.7, $\mathbf{H}$ is a strictly diagonally dominant matrix and has positive diagonal elements. So, $\lambda\mathbf{H}^T + (1-\lambda)\mathbf{H} + \mathbf{D}_1$ is strictly diagonally dominant when $|p| \leq \Delta$. It is well-known, under the assumptions $p(x) = 0$ and $q(x) \geq 0$, that $\mathbf{A}_3 = \mathbf{D}_2$ is strictly diagonally dominant and the diagonal elements of $\mathbf{D}_2$ are all positive. Thus $\mathbf{A} = \lambda\mathbf{H}^T + (1-\lambda)\mathbf{H} + \mathbf{D}_1 + \mathbf{D}_2$ generated by (4.5) is strictly diagonally dominant, which means the corresponding numerical scheme is stable. The proof is completed.

Combining Theorem 4.6, Theorem 4.7 and Theorem 4.8, we have following corollary.

Corollary 4.9. *Let $0 < \beta < 1$, $\lambda = 1/2$, $p(x) = 0$ and $q(x)$ be a nonnegative constant. Then the stiffness matrix $\boldsymbol{A}$ is a symmetric positive matrix.*

5 Numerical Examples

Example 1. Let $\beta = 1/2$ and $p(x) = q(x) = 1$. Applying Lemma 3.1, it can be verified that $u(x) = x^2 - x^3$ is the exact solution to the boundary value problem:

$$\begin{aligned} D(\lambda\ {}_0D_x^{-1/2} + (1-\lambda)\ {}_xD_1^{-1/2}) + Du + u &= f, \\ u(0) = 0,\ u(1) &= 0, \end{aligned}$$

where

$$f(x) = 2x - 2x^2 - x^3 + \frac{1}{\Gamma(1/2)}\left(\lambda f_1(x) + (1-\lambda) f_2(x)\right),$$

$f_1(x) = -4x^{\frac{1}{2}} + 8x^{\frac{3}{2}}, f_2(x) = -(1-x)^{-\frac{1}{2}} + 8(1-x)^{\frac{1}{2}} - 8(1-x)^{\frac{3}{2}}$.

As $u \in H_0^2(\Omega)$, Theorem 4.1 predicts a rate of convergence of 2 in the L^2 norm. Table 1 includes numerical results over a uniform partition of $[0,1]$, which support the predicted rates of convergence for different values of λ.

Table 1. Errors and convergence rates (Conv.) for different λ

h	$\Vert u-u_h\Vert_{L^2(\Omega)}$ $\lambda=1$	Conv. rate	$\Vert u-u_h\Vert_{L^2(\Omega)}$ $\lambda=0$	Conv. rate	$\Vert u-u_h\Vert_{L^2(\Omega)}$ $\lambda=1/2$	Conv. rate
1/4	1.0583E-2		1.0215E-2		7.8734E-3	
1/8	2.7836E-3	1.9267	2.6425E-3	1.9507	2.1589E-3	1.8667
1/16	6.8549E-4	2.0217	5.7384E-4	2.2032	5.0728E-4	2.0895
1/32	1.6731E-4	2.0346	1.4986E-4	2.1382	9.9524E-5	2.3497
1/64	4.1738E-5	2.0031	3.4030E-5	1.9376	2.3099E-5	2.1072
1/128	1.0286E-5	2.0206	7.9509E-6	2.0976	4.9756E-5	2.2149

Table 2 is the stiffness matrix $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3$ for $\lambda = 1$, $N = 8$, where $\mathbf{A}_1$ and $\mathbf{A}_2$ are calculated by (4.7) and (4.8), and $\mathbf{A}_3$ is computed by Gaussian quadrature rules. It is easy to verify that $\mathbf{A}$ is quasi-sparse, strictly diagonally dominated, and $\mathbf{A}$ satisfies some other special properties in Corollary 4.3, Theorems 4.4, 4.5, 4.7 and Theorem 4.8.

Example 2. We consider the 1-D form of the FADE that describes contaminant transport in groundwater flow [5].

$$\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial x} + \mathscr{D}\left(\lambda\frac{\partial^\mu C}{\partial x^\mu} + (1-\lambda)\frac{\partial^\mu C}{\partial(-x)^\mu}\right), \tag{5.1}$$

where $C(x,t)$ is the expected concentration, v is a constant mean velocity, x is distance in the direction of mean velocity, $\mathscr{D}$ is a constant dispersion coefficient, λ describes the skewness of transport process, $\frac{\partial^\mu C}{\partial x^\mu} = {}_0\mathrm{D}_x^\mu C$ and $\frac{\partial^\mu C}{\partial(-x)^\mu} = {}_x\mathrm{D}_1^\mu C$. A semi-discretization of FADE (5.1) using back finite differences in time is given by

$$\begin{aligned} &-\mathrm{D}(\lambda\ {}_0\mathrm{D}_x^{-\beta} + (1-\lambda)\ {}_x\mathrm{D}_1^{-\beta})\mathrm{D}u^{n+1} + p\mathrm{D}u^{n+1} + qu^{n+1} = f^{n+1}(x), \\ &n = 0, 1, 2\cdots \end{aligned} \tag{5.2}$$

where $u^{n+1}(x) = C(x,(n+1)\Delta t), \beta = 2-\mu$, $p = \frac{v}{\mathscr{D}}, q = \frac{1}{\mathscr{D}\Delta t}$, $f^{n+1}(x) = \frac{u^n(x)}{\mathscr{D}\Delta t}$ and Δt is the time-step size. Our numerical simulations take $\lambda = \frac{1}{2}$, $\Delta t = 0.02$, $v = 0.5$, $\mathscr{D} = 0.08$, and $\mu = 1.3$. We use δ distribution $\delta(x - \frac{1}{8})$ as the initial condition and the spatial meshsize h is taken as 0.01. Fig. 1 gives the plots when some parameters are changed. The behaviors of numerical solution $C(x,t)$ can be interpreted well in real physical cases (see e.g., [5], [6], [7]), which shows the effectiveness of FEM described in this paper.

Table 2. The stiffness matrix for $\lambda = 1$

+2.5761	-1.6069	0	0	0	0	0
-0.2290	+2.5761	-1.6069	0	0	0	0
-0.3957	-0.2290	+2.5761	-1.6069	0	0	0
-0.0927	-0.3957	-0.2290	+2.5761	-1.6069	0	0
-0.0413	-0.0927	-0.3957	-0.2290	+2.5761	-1.6069	0
-0.0228	-0.0413	-0.0927	-0.3957	-0.2290	+2.5761	-1.6069
-0.0142	-0.0228	-0.0413	-0.0927	-0.3957	-0.2290	+2.5761

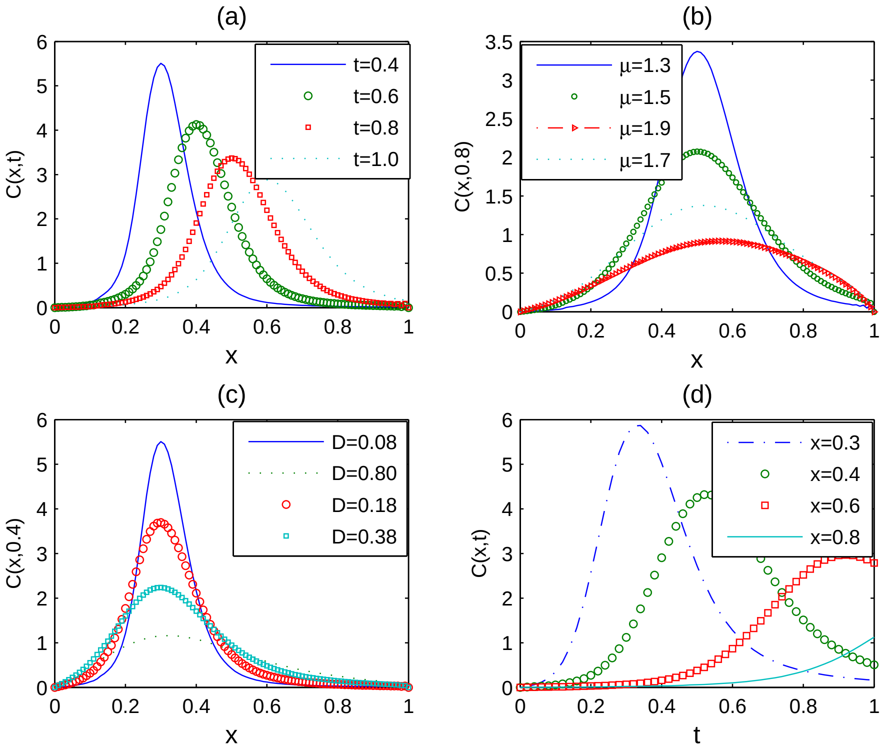

Fig 1. The evolution of $C(x,t)$: (a) $C(x,t)$ at different time; (b) $C(x,.)$ for different μ at $t = 0.8$; (c) $C(x,.)$ for different $\mathscr{D}$ at $t = 0.4$; (d) $C(.,t)$ for different x

6 Conclusions

FADE has a strong physical background, for example, contaminant transport in groundwater flow (see e.g., [5]). So far, it seems that there are no papers that take into account the detailed implementation of FEM. By selecting an appropriate variational formulation, we find that the stiffness matrix possesses some special properties, such as symmetry, sparseness and strictly diagonally domination, which greatly reduce the computational cost and storage requirement and guarantee the stability of finite element equations.

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Competing Interests

Authors have declared that no competing interests exist.

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