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A Survey on Kannan Mappings in Universalized Metric Spaces

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Authors' contributions

This work was carried out in collaboration between all authors. First author AM is a supervisor, second author NY and third author BM are students of him. The problems are given to the student s from by first author AM and second author NY wrote the first draft of the manuscript. First author AM and third author BM have managed and edited the manuscript. All authors read and approved the final manuscript.

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Abstract

In this manuscript, firstly, Kannan contraction type mapping is universalised. Secondly, the unique fixed point of the universalized Kannan contraction type mappings in universalized metric space is verified. Furthermore, we clarify that it can not have a fixed point in metric space.

Keywords: Universalized metric space; Universalized Kannan contraction type mappings; fixed point.

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1 Introduction

The idea of a universalized metric space is acquainted by Branciari [1] where the triangle inequality of a metric space has been substituted by an inequality implying three terms instead of two.

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The well-known Banach fixed point theorem maintained that if (X,d) is a complete metric space and $T: X \to X$ is a map such that for each $x, y \in X$

$$d(Tx,Ty) \leq kd(x,y)$$

where $0 \leq k < 1$, then T has a unique fixed point $\overline{x} \in X$ and for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to $\overline{x}$.

The Banach contraction principle was enhanced in such spaces by Branciari [1]. In addition, Banach contraction mapping theorem was formed in universalized metric space. Furthermore, he provided an example to demonstrate there exist universalized metric space which is not metric space. In [2-4] some other fixed point conclusions were constituted in universalized metric spaces.

The concept of $G-$ Kannan maps were described and obtanied a fixed point theorem for such mappings in [5]. It is well known that fixed point results for Kannan type mappings were extended to $K-$ metric spaces and a universalization of Kannan's fixed point theorem was given to the spaces in [6]. The unique common fixed point of Kannan type mapping was examined on completed metric spaces based on a function and satisfied the sufficent connditions in [7]. The Kannan type contractions was universalized and extended with some auxiliary functions to obtain some new fixed point results in the framework of $b-$ metric spaces in [8]. In [9,10], the type Kannan fixed point theorem was verified in universalized metric space. The existence of a unique fixed point of T-Kannan type mappings on complete cone metric spaces and some fixed point theorems were extended and generalized in [11].

Recently, several of universalizations of the above Banach contraction principle have occured. One of all these is the undermentioned universalization of Kannan contraction type mapping.

Theorem 1 [4] If (Y,ρ) is a complete metric space and $T: Y \to Y$ satisfies

$$\rho(Tx,Ty) \leq \beta[\rho(x,Tx) + \rho(y,Ty)] \quad (1)$$

for all $x, y \in Y$, where $0 < \beta < \frac{1}{2}$, then T has a unique fixed point in Y.

Definition 2 [4] Let (X,d) be a universalized metric space. The self-map $T: X \to X$ is named a universalized Kannan contraction type map if

$$d(Tx,Ty) \leq \alpha(d(x,y))[d(x,Tx) + d(y,Ty)] \quad (2)$$

for any $x, y \in X$, where $\alpha: [0,\infty] \to [0,\frac{1}{2})$ is a increasing mapping.

2 Main Consequence

Suppose X is not a null set and $d: X \times X \to [0,\infty]$ is a mapping. If d appeases all of the commonly cases of a metric except that the value of d may be infinity, we claim that (X,d) is a universalized metric space.

Now we define the notion of a universalized Kannan contraction type map in universalized metric spaces.

Now the main conclusion is certified.

Theorem 3 Assume (X,d) is a universalized metric space. Let $T:X\to X$ be a universalized Kannan contraction type mapping such that α satisfies for each $r\in[0,\infty)$

$$\limsup_{t\to r}\alpha(t)<\frac{1}{2}. \tag{3}$$

Suppose that there exist an $x_0\in X$ with the bounded circumgyration, that is, the sequence $\{T^n x_0\}$ is bounded. Additionally, assume that $d(x,Tx)<\infty$ for each $x\in X$. Then T has a fixed point $\tilde{x}\in X$ and $\lim_{n\to\infty}T^n x_0=\tilde{x}$. Besides, if T has a fixed point $\tilde{y}$, then either $d(\tilde{x},\tilde{y})=\infty$ or $\tilde{x}=\tilde{y}$.

Proof. Take $x_0\in X$ be arbitrary. Describe a sequence $\{x_n\}$ in the following way:

$$x_{n+1}=T^n x_0 \quad n=0,1,3,\cdots. \tag{4}$$

From T is a universalized Kannan contraction type mapping, we get

$$d(T^n x_0,T^{n+1}x_0)=d(T(T^{n-1}x_0),T(T^n x_0))$$

$$\leq\alpha(d(T^{n-1}x_0,T^n x_0))[d(T^{n-1}x_0,T^n x_0)+d(T^n x_0,T^{n+1}x_0)].$$

Then

$$d(T^n x_0,T^{n+1}x_0)\leq\frac{\alpha(d(T^{n-1}x_0,T^n x_0))}{1-\alpha(d(T^{n-1}x_0,T^n x_0))}d(T^{n-1}x_0,T^n x_0).$$

Since $\alpha(t)\in[0,\frac{1}{2})$, we get $\frac{\alpha(t)}{1-\alpha(t)}<1$. Then we get

$$d(T^n x_0,T^{n+1}x_0)<d(T^{n-1}x_0,T^n x_0).$$

It follows that $\{d(T^n x_0,T^{n+1}x_0)\}$ is monotone decreasing.

Similarly we can show following statement

$$d(T^{n-1}x_0,T^n x_0)\leq\frac{\alpha(d(T^{n-2}x_0,T^{n-1}x_0))}{1-\alpha(d(T^{n-2}x_0,T^{n-1}x_0))}d(T^{n-2}x_0,T^{n-1}x_0).$$

Because $\alpha(t)$ is increasing, $\frac{\alpha(t)}{1-\alpha(t)}$ also is increasing. Furthermore, from $\{d(T^{n-2}x_0, T^{n-1}x_0)\}$ is monotone decreasing, then $d(T^{n-2}x_0, T^{n-1}x_0) < d(x_0, Tx_0)$. Hence

$$\frac{\alpha(d(T^{n-2}x_0, T^{n-1}x_0))}{1-\alpha(d(T^{n-2}x_0, T^{n-1}x_0))} < \frac{\alpha(d(x_0, Tx_0))}{1-\alpha(d(x_0, Tx_0))}.$$

Then

$$d(T^{n-1}x_0, T^{n}x_0) \le \frac{\alpha(d(x_0, Tx_0))}{1-\alpha(d(x_0, Tx_0))} d(T^{n-2}x_0, T^{n-1}x_0).$$

Repeating this relation we get

$$d(Tx_0, T^{2}x_0) \le \frac{\alpha(d(x_0, Tx_0))}{1-\alpha(d(x_0, Tx_0))} d(x_0, Tx_0).$$

Now let $h = \frac{\alpha(d(x_0, Tx_0))}{1-\alpha(d(x_0, Tx_0))}$, then we have

$$d(T^{n-1}x_0, T^{n}x_0) \le h\ d(x_0, Tx_0). \tag{5}$$

Consequently, we obtain from (5) for $m > n$;

$$\begin{aligned}
&d(T^{n}x_0, T^{m}x_0) \le d(T^{n}x_0, T^{n+1}x_0) \\
&+d(T^{n+1}x_0, T^{n+2}x_0) + \cdots + d(T^{m-1}x_0, T^{m}x_0) \\
&\le (h)^{n} d(x_0, Tx_0) + (h)^{n+1} d(x_0, Tx_0) + \cdots + (h)^{m-1} d(x_0, Tx_0) \\
&= [(h)^{n} + (h)^{n+1} + \cdots + (h)^{m-1}] d(x_0, Tx_0) \\
&\le \frac{(h)^{n}}{1-h} d(x_0, Tx_0)
\end{aligned}$$

Since $h = \frac{\alpha(d(x_0, Tx_0))}{1-\alpha(d(x_0, Tx_0))} \in [0,1)$, it follows that $\{T^{n}x_0\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $\tilde{x} \in X$ such that $T^{n}x_0 \to \tilde{x}$.

Now let us show that T has a fixed point $\tilde{x}$. To illustrate this ascertain we display that there exist $0 < k < \frac{1}{2}$ such that $\alpha(d(\tilde{x}, T^{n}x_0)) < k$ for each $n \in \mathbb{N}$. Contrary to ordinary, assume that $\lim_{j\to\infty} d(\tilde{x}, T^{n_j}x_0) = \frac{1}{2}$ for some subsequence n_j. Since $\lim_{j\to\infty} d(\tilde{x}, T^{n_j}x_0) = 0$, then from the above, we get $\limsup_{t\to\infty^+}\alpha(t)=\frac{1}{2}$, a contradiction. Since T is a universalized Kannan contraction type map, then we have

$$\begin{aligned} d(T\tilde{x},T^{n+1}x_0) &\le \alpha(d(\tilde{x},T^n x_0))\left[d(\tilde{x},T\tilde{x})+d(T^n x_0,T^{n+1}x_0)\right] \\ &\le k\left[d(\tilde{x},T\tilde{x})+d(T^n x_0,T^{n+1}x_0)\right] \end{aligned}$$

Taking the limit as $n\to\infty$, yields

$$d(T\tilde{x},\ \tilde{x})=\limsup_{n\to\infty} d(T\tilde{x},T^{n+1}x_0)\le kd(\tilde{x},\ T\tilde{x}),$$

which yields $d(T\tilde{x},\ \tilde{x})=0$, and so $T\tilde{x}=\tilde{x}$. Consider that T has two fixed points $\tilde{x}$ and $\tilde{y}$ such that $d(\tilde{x},\tilde{y})<\infty$. Then

$$d(\tilde{x},\tilde{y})=d(T\tilde{x},T\tilde{y})\le \alpha(d(\tilde{x},\tilde{y}))\left[d(\tilde{x},\ T\tilde{x})+d(\tilde{y},T\tilde{y})\right].$$

Since $\alpha(d(\tilde{x},\tilde{y}))<\frac{1}{2}$, so $\tilde{x}=\tilde{y}$.

The following simple example shows Theorem 3 is not true in metric spaces if we assume α is continuous and increasing.

Example 4 Take $X=(0,\infty)$ with the standard metric, $T:X\to X$ be given by $Tx=\frac{x}{4}$. Describe $\alpha:[0,\infty)\to[0,\frac{1}{2})$ by $\alpha(t)=\frac{t}{2+2t}$. Then, obviously, α is continuous and increasing, and

$$|Tx-Ty|\le\alpha(|x-y|)[|x-Tx|+|y-Ty|].$$

for each $x,y\in X$, but T has no fixed point in X.

Example 5 Take $X=\{0,1,\infty\}$, if $x=y$, then $d(x,y)=0$, and if $x\neq y$, then $d(x,y)=\frac{1}{2}$. Let $T:X\to X$ be given by $Tx=1$. Describe $\alpha:[0,\infty)\to[0,\frac{1}{2})$ by $\alpha(t)=t$. Then T provide condition of Theorem 3. Consequently T has a fixed point in X.

3 Conclusion

We have specified Kannan's fixed point theorems in universalized metric. It is interesting that each fixed point theorems are verifying in the theory, because the theory is consisting and are given examples.

Competing Interests

Authors have declared that no competing interests exist.

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