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$T_{s\hat{g}}$-space in Topological Spaces

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

The aim of this paper is to introduce and study the class of $s\hat{g}$-closed sets, which is properly placed between the classes of semi-closed sets due to Crossley and Hildebrand in 1974 and gs-closed sets due to Arya and Nour in 1990. Also, we investigated the relations of anew notion and the other notions of generalized closed. Moreover as applications, using the notion of $s\hat{g}$-closed sets, we introduce a new space called $T_{s\hat{g}}$-space.

Keywords: Generalized closed sets; $\hat{g}$ -closed sets; sg-closed sets; gs-closed sets; spg-closed sets; gsp-closed sets and $s\hat{g}$ -closed sets.

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1 Introduction

Closed sets are fundamental objects in a topological space. For example, one can define the topology on a set by using either the axioms for the closed sets or the Kuratowski closure axioms. In 1970, Levine [1] initiated the study of so-called generalized closed sets. By definition, a subset S of a topological space (X,τ) is called generalized closed if the closure of any subset A of X is included in every open superset of A. This

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notion has been studied extensively in recent years by many topologists because generalized closed sets are not only natural generalizations of closed sets. More importantly, they also suggest several new properties of topological spaces. Most of these new properties are separation axioms weaker than T_1, some of which have been found to be useful in computer science and digital topology. For example, the well known digital line is $T_{\frac{3}{4}}$ but not T_1. Other new properties are defined by variations of the property of submaximality. Furthermore, the study of generalized closed sets also provides new characterizations of some known classes of spaces, for example, the class of extremely disconnected spaces. In 1987, P. Bhattacharyya and B. K. Lahiri [2] introduced a new class of sets called semi-generalized closed sets by means of semi-open sets of N. Levine [3] and obtained various properties corresponding to [1]. In 1990, S. P. Arya and T.M. Nour [4] defined the generalized semi-closed sets Dontchev [5] introduced gsp-closed sets by generalizing semi-preopen sets. Recently, Veera Kumar [6] introduced ĝ -closed sets in topological spaces. In the present paper, we introduce a new class of closed sets called sĝ-closed sets and find some basic of its properties. We also prove that this class lies between the class of semi-closed sets and the class of gs-closed sets. Applying there sets we introduce a new space called $T_{s\hat{g}}$ -space.

2 Preliminaries

Throughout this paper (X, τ) and (Y, σ), represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$, $int(A)$ and A^cor $X\backslash A$ denote the closure of A, the interior of A and the complement of A in X , respectively .

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset A of a space (X, τ) is called

(a) Semi-closed [7] if $int\ (cl(A)) \subseteq A$,
(b) Semi-open [3] if $X\backslash A$ is semi-closed, or equivalently, if $A \subseteq cl\ (int((A))$,
(c) Semi-preclosed [8] (= β-closed [9]) if $int(cl(int(A))) \subseteq\ A$,
(d) Semi-preopen [8] (=β -open [9]) if $X\backslash A$ is semi-preclosed (= β-closed), or equivalently, if $A \subseteq cl\ (int(cl\ (A)))$.

The semi-closure [10,7] of a subset A of (X τ), denoted by $scl_X(A)$, briefly $scl(A)$, is defined to be the intersection of all semi-closed sets containing A. The semi-interior of A [7], denoted by $sint(A)$, is defined by the union of all semi-open sets contained in A.

Definition 2.2. A subset A of a space (X, τ) is said to be:

(a) Generalized closed [1] (briefly, g-closed)if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(b) Generalized semi-closed [4] (briefly, gs-closed)if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(c) Semi-generalized closed [2] (briefly, sg-closed)if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ).
(d) Generalized semi-preclosed [5] (briefly, gsp-closed) if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(e) ĝ-closed [6] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ). The complement of ĝ-closed set is called ĝ-open
(f) Semi-pre-generalized closed [11] (briefly, spg-closed) if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-preopen in (X, τ).

Definition 2.3. A topological space (X, τ) is called $T_{\hat{g}}$ -space [6] if every ĝ-closed set is closed.

3 $s\hat{g}$ -closed sets and Basic Properties

In this section, we introduce the concept of s$\hat{g}$-closed sets and study some of their properties and relations with other known classes of subsets.

Definition 3.1. A subset A of a space (X, τ) is called s$\hat{g}$-closed if $scl(A) \subseteq G$ whenever $A \subseteq G$ and G is a $\hat{g}$-open set in (X, τ).

Proposition 3.1. Every semi-closed set is s$\hat{g}$-closed in (X, τ).

Proof. Let A be a semi-closed set and G be any $\hat{g}$-open set containing A. Since A is semi-closed, $scl(A) = A$ for every subset A of X. Therefore, $scl(A) \subseteq G$ and hence A is s$\hat{g}$-closed.

Corollary 3.1. Every closed set is s$\hat{g}$-closed in (X, τ).

Remark 3.1. The following example shows that the converse of Proposition 3.1, need not be true.

Example 3.1. Let $X = \{a, b, c\}$ with a topology $\tau = \{X, \emptyset, \{a, b\}\}$. Then the set $\{a, c\}$ is s$\hat{g}$-closed but not semi-closed in (X, τ).

Proposition 3.2. Every s$\hat{g}$-closed set is gs-closed in (X, τ).

Proof. Let A be s$\hat{g}$-closed and G be any open set containing A in (X, τ). Since every open set is $\hat{g}$-open, the proof follows immediately.

Remark 3.2. The converse of the above proposition need not be true in general as shown by the following example.

Example 3.2. Let τ be the usual topology on the real line R. The open interval (a, b) is gs-closed but not s$\hat{g}$-closed.

From the above results we note that the class of s$\hat{g}$-closed sets lies between the class of semi-closed sets and the class of sg-closed sets.

Proposition 3.3. Every s$\hat{g}$-closed set is gsp-closed in (X, τ).

Proof. It is true that $spcl(A) \subseteq scl(A)$ for every subset A of X.

Remark 3.3. A gsp-closed set need not be s$\hat{g}$-closed as seen from the following example.

Example 3.3. Let $X = \{a, b, c\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ It is easy to check that the set $\{a\}$ is gsp-closed but not s$\hat{g}$-closed in (X, τ).

Remark 3.4. The following examples show that s$\hat{g}$-closedness is independent from spg-closedness, g-closedness, $\hat{g}$-closedness and sg-closedness

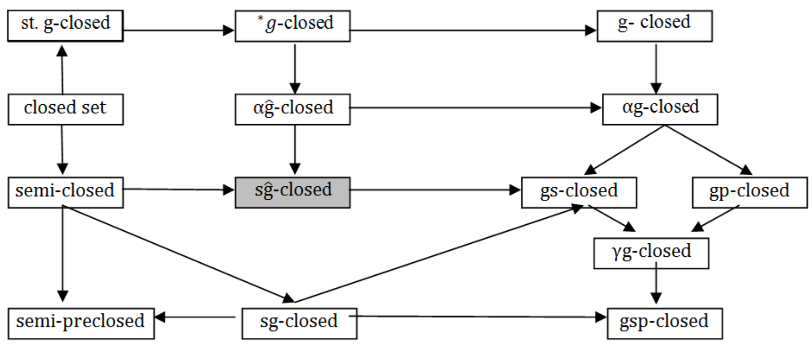
Example 3.4. Let $X = \{a, b, c\}$ and τ = $\{X, \emptyset, \{a\}, \{a, c\}\}$. Then a set:

(a) $A = \{a, b\}$ is s$\hat{g}$-closed but not spg-closed in (X, τ).
(b) $B = \{c\}$ is s$\hat{g}$-closed but neither g-closed nor $\hat{g}$-closed in (X, τ).
(c) $C = \{a\}$ in Example 3.3. is spg-closed but not s$\hat{g}$-closed.
(d) $D = \{b\}$ in Example 3.3. is g-closed , $\hat{g}$-closed and sg-closed but not s$\hat{g}$-closed.

Example 3.5. Let $X = \{a, b, c\}$ with topology $\tau = \{X, \emptyset, \{a\}\}$. Then the set $\{a, b\}$ is s$\hat{g}$-closed but not sg-closed.

Example 3.6. Let X be the real line and let τ be the point generated topology on X, i.e. the non-empty open sets are those containing a fixed point, say the zero point. Then the set P of all irrationals is closed in (X, τ) and thus s$\hat{g}$-closed. Since the semi- regularization $(X, \tau_s)$is the indiscrete space, P is gs-closed in (X, τ_s) but not spg-closed in a space (X, τ) .

Remark 3.5. From the above discussions and known results we have the following implications.



Theorem 3.1. If A and B are s$\hat{g}$-closed sets in (X, τ), then $A \cup B$ is s$\hat{g}$-closed in (X, τ).

Proof. Follows directly from the observation $scl(A \cup B) = scl(A) \cup scl(B)$ for all $A, B \subset X$.

Theorem 3.2. The finite union of s$\hat{g}$-closed sets is s$\hat{g}$-closed in (X, τ).

Proof. Let $\{F_i: i = 1,2, \dots, n\}$ be a finite class of s$\hat{g}$-closed subsets of a space (X, τ) , then for each $\hat{g}$-open set U_i in X containing F_i and $scl(F_i) \subseteq U_i, i \in \{1,2, \dots, n\}$. Hence $\bigcup_i F_i \subseteq \bigcup_i U_i = V$. Since an arbitrary union of $\hat{g}$-open sets in (X, τ) is also $\hat{g}$-open ,V is $\hat{g}$-open in (X, τ) .Also $\bigcup_i scl(F_i) = scl(\bigcup_i F_i) \subseteq V$. Therefore, $\bigcup_i F_i$ is s$\hat{g}$- closed in (X, τ).

Remark 3.6. Intersection of any two is s$\hat{g}$-closed sets in (X, τ) may fail to be s$\hat{g}$-closed, as the following example shown.

Example 3.7. Consider a topological space (X, τ) as in Example 3.5. The sets $\{a, b\}$ and $\{a, c\}$ are s$\hat{g}$-closed but their intersection $\{a\}$ is not an s$\hat{g}$-closed subset of X.

Proposition 3.4. If a set A is s$\hat{g}$-closed, then $scl(A)\backslash A$ contains no, non-empty closed set.

Proof. Suppose that A is s$\hat{g}$-closed. Let F be a closed subset of $(A)\backslash A$. Then $A \subseteq F^c$, F^c is open and hence $\hat{g}$-open. Since A is s$\hat{g}$-closed, $scl(A) \subseteq F^c$. Consequently, $F \subseteq (scl(A))^c$. Every closed set is semi-closed and hence F is semi-closed. Therefore, $F \subseteq scl(A)$. Hence $F \subseteq scl(A) \cap (scl(A))^c$ and hence F is empty.

Remark 3.7. The converse of the above proposition need not be true. Consider a topological space (X, τ) as in Example 3.2. If $A = \{b\}$, then $sc(A)\backslash A = \{c\}$ does not contain non-empty closed set. However, A is not s$\hat{g}$-closed in (X, τ) .

Proposition 3.5. If a set A is s$\hat{g}$-closed in (X, τ), then $scl(A)\backslash A$ does not contain any non-empty $\hat{g}$-closed set.

Proof. Assume that A is s$\hat{g}$-closed, let F be a $\hat{g}$-closed set contained in $scl(A)\backslash A$. Since F^c is $\hat{g}$-open set with $A \subseteq F^c$ and A is s$\hat{g}$-closed, $scl(A) \subseteq F^c$. That is $F \subseteq (scl(A))^c$. Also $F \subseteq scl(A)\backslash A$. Therefore, $F \subseteq (scl(A))^c \cap scl(A) = \emptyset$. Hence F =$\emptyset$.

Proposition 3.6. If A is $\hat{g}$-open and s$\hat{g}$-closed subset of (X, τ), then A is semi-closed subset of (X, τ).

Proof. Since A is $\hat{g}$-open and s$\hat{g}$-closed, $scl(A) \subseteq A$. Hence A is semi-closed.

Theorem 3.3. In a T_1 space every s$\hat{g}$-closed set is semi-closed.

Proof. Let X be a T_1-space nd $A \subseteq X$ be an s$\hat{g}$-closed set. Let $x \in A,\ y \in scl(A)\backslash A$ and $x \neq y$, then there exists a $\hat{g}$-open set $U_x \subseteq X$ such that $x \in U_x$ and $y \notin U_x$. This implies $A \subseteq \bigcup_{x\in A} U_x = V, y \notin V$. Since A is s$\hat{g}$-closed, $scl(A) \subseteq V$ and $scl(A)\backslash A = \emptyset$. So $A = scl(A)$ and therefore A is semi-closed.

Proposition 3.7. Let A be a s$\hat{g}$-closed subset of (X, τ). If $A \subseteq B \subseteq scl(A)$, then B is also an s$\hat{g}$-closed subset of (X, τ).

Proof. Let U be a $\hat{g}$-open set of (X, τ) such that $B \subseteq U$, then $A \subseteq U$. Since A is an s$\hat{g}$-closed set, $scl(A) \subseteq U$. Also, since $B \subseteq scl(A)$, $scl(B) \subseteq scl(scl(A)) = scl(A)$. Thus, $scl(B) \subseteq U$. Hence B is also an s$\hat{g}$-closed subset of (X, τ).

Proposition 3.8. Let $A \subseteq Y \subseteq X$ and let A be s$\hat{g}$-closed in X. Then A is s$\hat{g}$-closed relative to Y.

Proof. Let $A \subseteq Y \cap G$ and suppose that G is $\hat{g}$-open in X, then $A \subseteq G$. Since A is s$\hat{g}$-closed in X, $scl(A) \subseteq G$. It follows that $Y \cap scl(A) \subseteq Y \cap G$. Thus, $scl_Y(A) \subseteq Y \cap G$. Hence A is s$\hat{g}$-closed relative to Y.

Proposition 3.9. For each $x \in X$ either $\{x\}$ is $\hat{g}$-closed or $\{x\}^c$ is s$\hat{g}$-closed set in X.

Proof. Suppose that {x} is not $\hat{g}$-closed in X . Then $\{x\}^c$ is not $\hat{g}$-open and the only $\hat{g}$-open set containing $\{x\}^c$is the space X itself. That is $\{x\}^c \subseteq$ X. Therefore, $scl(\{x\}^c) \subseteq X$ and so $\{x\}^c$ is s$\hat{g}$-closed.

Theorem 3.4. Let A be s$\hat{g}$-closed in X. Then A is semi-closed if and only if $scl(A)\backslash A$ is closed.

Proof. Necessity . Let A be a semi-closed subset of X. Then $scl(A) = A$ and so, $scl(A)\backslash A = \emptyset$ which is closed.

Sufficiency. Since A is s$\hat{g}$-closed, by Proposition 3.4, $scl(A)\backslash A$ contains no non - empty closed set, but $scl(A)\backslash A$ is closed. This implies $scl(A)\backslash A = \emptyset$. That is $scl(A) = A$. Hence A is semi-closed.

4 s$\hat{g}$-open Sets

Definition 4.1. A subset A of a space (X, τ) is called s$\hat{g}$-open if A^c is s$\hat{g}$-closed in (X, τ).

Remark 4.1. For a subset A of (X, τ), $scl(A^c) = (sint(A))^c$

Theorem 4.1. $A \subseteq X$ is s$\hat{g}$-open if and only if $F \subseteq sint(A)$ whenever F is $\hat{g}$-closed and $F \subseteq A$.

Proof. Necessity. Let A be a sĝ-open set in (X, τ). Let F be ĝ-closed such that $F \subseteq A$. Then $A^c \subseteq F^c$ where F^c is ĝ-open. A^c is sĝ-closed implies that $scl(A^c) \subseteq F^c$. By Remark 4.1, $(sint(A))^c \subseteq F^c$. That is $F \subseteq sint(A)$.

Sufficiency. Suppose F is ĝ-closed and $F \subseteq A$ implies $F \subseteq sint(A)$. Let $A^c \subseteq U$ where U is ĝ-open. Then $U^c \subseteq A$ where U^c is ĝ-closed. By hypothesis $U^c \subseteq sint(A)$. That is $(sint(A))^c \subseteq U^c$. By Remark 4.1, $scl(A^c) \subseteq U$. This implies A^c is sĝ-closed. Hence A is sĝ-open.

Proposition 4.1. If $sint(A) \subseteq B \subseteq A$ and A is sĝ-open, then B is sĝ-open.

Proof. $sint(A) \subseteq B \subseteq A$ implies $A^c \subseteq B^c \subseteq (sint(A))^c$. By Remark 4.1, $A^c \subseteq B^c \subseteq sint(A^c)$.Also, A^c is sĝ-closed . By Proposition 3.7, B^c is sĝ-closed. Hence B is sĝ-open.

Remark 4.2. It is true that every open set is sĝ-open but the converse may not be true as seen in the following example.

Example 4.1. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. In (X, τ), the set $\{b\}$ is sĝ-open but not open.

As application of the concept of sĝ-closed sets we introduce the following space called $T_{s\hat{g}}$ -space.

Definition 4.2. A topological space (X, τ), is called a $T_{s\hat{g}}$ -space if every sĝ-closed set is closed in X.

Remark 4.3. The space $T_{s\hat{g}}$ and $T_{\hat{g}}$ are independent as is seen from the following examples.

Example 4.2. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a\}\}$. Then the space (X, τ)is a $T_{\hat{g}}$-space but not a $T_{s\hat{g}}$ -space.

Example 4.3. Let X ={a,b,c} with τ= {∅,{a},{b,c},X}. Then (X, τ) is $T_{s\hat{g}}$ -space but it is not $T_{\hat{g}}$ -space.

5 Conclusion

Generalizations of closed sets in point-set topology will give some new topological properties (for example, separation axioms, compactness, connectedness, continuity) which have been found to be very useful in the study of certain objects of digital topology. Thus we may stress once more the importance of sĝ-closed sets as a branch of them and the possible application in computer graphics [12-14] and quantum physics [15].

Competing Interests

Author has declared that no competing interests exist.

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