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Theory of Hypersurfaces Y^{n-1} in Y^n Space and Geodesics on this Hypersurfaces Y^{n-1}

Yaremenko Mikola (Nikolay) Ivanovich[1,2,3*]

[1]Department of Advance Mathematics, National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Kyiv 03056, Solomenskiy District, Ave. Pobedy, 37, Ukraine.
[2]Institute of Mathematics of the National Academy of Sciences of Ukraine, Kiev 010044, Tereschenkivska St. 3, Ukraine.
[3]Department of Mathematics, Claremont Graduate University, USA.

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

The hypersurface Y^{n-1} in Y^n space is studied for this piece of work. We established the correlation between tensors of hypersurface Y^{n-1} and tensors of embedding space Y^n . The second non-symmetrical tensor of hypersurface has been introduced, which have been obtained from the analog of Peterson-Codazzi equation in nonsymmetrical case.Also we have introduced the tensor that is associated with square of angle between normal and adjacent normal and it is represented in terms of metric and second tensors of hypersurface. The geodesics on hypersurface have been studied, and nontrivial example of geodesics on hypersurface with torsion and Euclid metric was constructed.

Keywords: Hypersurface; gravity; geometry; geodesic; torsion; second tensor of hypersurface; connection.

Mathematics Subject Classification: 53B15; 53A40; 53A99; 53B50; 53B99.

**Corresponding author: E-mail: math.kiev@gmail.com;*

1 Introduction

The hypersurface theory in the Y^n space has attracted a lot of attention after its introduction [1,2], since it has many applications in theoretical and applied physics and electromagnetic gravity.

Especially interesting is a special case of three-dimensional hypersurface Y^3 that embedding in four-dimensional space Y^4, which can be associated with physical spacetime continuum.

Intrinsic geometrical properties have been considered in [1-9] a space and their context and the relations with the ambient space Y^n within [9].

Riemannian geometry is considered smooth manifolds with a Riemannian metric without torsion [10]; this is a concept of distance which is given by means of a smooth positive definite symmetric tensor defined at each point. Many works have been done to generalize the concept of Riemannian manifolds [11-14,15-17, [1]. It is obvious that a smooth manifold always carries a natural vector bundle, that is called a tangent bundle, this structure can be sufficient for construction of analysis on the manifold, however, to consider geometrical properties requires to have relation between points of space at different points that is called a parallel transport of geometrical objects that induced the concept of connection in space. The concept of connection gives a possibility to studied geometrical futures that demanded to move some object in space, for instance, geodesic lines [9]. In this article, we consider the geometrical properties of hypersurfaces, which embedding in space, that approach gives us possibilities to introduce the concepts of nonsymmetrical second tensor of hypersurface and found the representation of the tensor, which is associated with square of angle between normal and adjacent normal in terms of metric and second tensors of hypersurface. In this article, it is assumed that hypersurfaces Y^{n-1} is given as natural subspace of Y^n space and not forcibly emerged in high dimensions spaces.

The remainder of this article is organized as follows. Section 2 discusses the geometrical structure of hypersurface and its correlation with geometrical structure of embedding space; the analogue of Peterson-Codazzi equation in the nonsymmetrical case has been obtained. The geodesics on hypersurface are presented in Section 3, which is followed in Section 4 by the discussion of the nontrivial application of the example of such geodesics on hypersurface space with torsion and Euclid metric and deduction the equation of these geodesics lines. Section 5 provides some final conclusions and directions for future work.

2 The Hypersurfaces Y^{n-1} in the Y^n Space

We are studying the hypersurfaces Y^{n-1} in the space with torsion Y^n. Let us assumed that the hypersurface can be defined by a system of equations

$$x^i = x^i\left(y^1,\dots,y^{n-1}\right),$$

where x^i is a coordinate system in Y^n space, y^α is a coordinate system in Y^{n-1} subspace and the rank of the matrix $\left[\frac{\partial x^i}{\partial y^\alpha}\right]$ equal $n-1$. The metric tensor of a hypersurface Y^{n-1} is given by:

$$a_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}. \tag{1}$$

Let $G^{\alpha}_{\beta\gamma}$ is a geometric object and is subjected to the law of the transformation from one coordinate system y^{α} to another hachure coordinate system $y^{\alpha'}$ by the formula:

$$G^{\alpha}_{\beta\gamma}=G^{\alpha'}_{\beta'\gamma'}\frac{\partial y^{\alpha}}{\partial y^{\alpha'}}\frac{\partial y^{\beta'}}{\partial y^{\beta}}\frac{\partial y^{\gamma'}}{\partial y^{\gamma}}+\frac{\partial y^{\alpha}}{\partial y^{\alpha'}}\frac{\partial^2 y^{\alpha'}}{\partial y^{\beta}\partial y^{\gamma}}. \qquad (2)$$

Then we assume that connection $G^{\alpha}_{\beta\gamma}$ of is is associated with the connection Γ^{k}_{ij} of Y^{n} mean of the formula:

$$G^{\alpha}_{\beta\gamma}\frac{\partial x^k}{\partial y^{\alpha}}=\Gamma^{k}_{ij}\frac{\partial x^i}{\partial y^{\beta}}\frac{\partial x^j}{\partial y^{\gamma}}+\frac{\partial^2 x^k}{\partial y^{\beta}\partial y^{\gamma}}. \qquad (3)$$

We are obtaining

$$\frac{1}{2}\left(a^{\alpha\eta}\left(a_{\beta\eta,\gamma}+a_{\gamma\eta,\beta}+a_{\beta\gamma,\eta}+a_{\beta\mu}T^{\mu}_{\gamma\eta}+a_{\gamma\mu}T^{\mu}_{\beta\eta}\right)+T^{\alpha}_{\gamma\beta}\right)\frac{\partial x^k}{\partial y^{\alpha}}=$$
$$=\frac{1}{2}(g^{kn}\left(g_{ni,j}+g_{nj,i}-g_{ij,n}+g_{im}S^{m}_{jn}+g_{jm}S^{m}_{in}\right)+S^{k}_{ij}\,)\frac{\partial x^i}{\partial y^{\beta}}\frac{\partial x^j}{\partial y^{\gamma}}+\frac{\partial^2 x^k}{\partial y^{\beta}\partial y^{\gamma}}. \qquad (4)$$

By permuting indices, we have next formula for the torsion tensor of hypersurface Y^{n-1}:

$$T^{\gamma}_{\alpha\beta}=a^{\gamma\eta}g_{pq}S^{p}_{ij}\frac{\partial x^i}{\partial y^{\alpha}}\frac{\partial x^j}{\partial y^{\beta}}\frac{\partial x^q}{\partial y^{\eta}} \qquad (5)$$

using tensors $a_{\alpha\beta}$ and $T^{\gamma}_{\alpha\beta}$ both metric and torsion we can explore the geometry of the space hypersurface Y^{n-1}.

So, we obtained the formula for a tensor of torsion $T^{\gamma}_{\alpha\beta}$ of hypersurface Y^{n-1} (assuming that functions $x^i(y^1,\dots,y^{n-1})$ are smooth enough) in form $T^{\gamma}_{\alpha\beta}=a^{\gamma\eta}g_{pq}S^{p}_{ij}\frac{\partial x^i}{\partial y^{\alpha}}\frac{\partial x^j}{\partial y^{\beta}}\frac{\partial x^q}{\partial y^{\eta}}$. Let $G^{\alpha}_{\beta\gamma}$ be the connection of Y^{n-1} and we assume that $G^{\alpha}_{\beta\gamma}$ express via metric $a_{\alpha\beta}$ and torsion $T^{\gamma}_{\alpha\beta}$ similarly to as the connection Γ^{k}_{ij} express by means of and, we have:

$$G^{\alpha}_{\beta\gamma}=\frac{\mathbf{1}}{\mathbf{2}}\left(a^{\alpha\eta}\left(a_{\beta\eta,\gamma}+a_{\gamma\eta,\beta}+a_{\beta\gamma,\eta}+a_{\beta\mu}T^{\mu}_{\gamma\eta}+a_{\gamma\mu}T^{\mu}_{\beta\eta}\right)+T^{\alpha}_{\gamma\beta}\right). \qquad (6)$$

From (3), we have

$$\left(G^{\alpha}_{\beta\gamma}-G^{\alpha}_{\gamma\beta}\right)\frac{\partial x^k}{\partial y^{\alpha}}=\left(\Gamma^{k}_{ij}-\Gamma^{k}_{ji}\right)\frac{\partial x^i}{\partial y^{\beta}}\frac{\partial x^j}{\partial y^{\gamma}}+\left(\frac{\partial^2 x^k}{\partial y^{\beta}\partial y^{\gamma}}-\frac{\partial^2 x^k}{\partial y^{\gamma}\partial y^{\beta}}\right),$$

if we assume that functions $x^i(y^1,\dots,y^{n-1})$ have enough smoothness, the last difference equals to zero; and we obtain

$$T^{\alpha}_{\beta\gamma}\frac{\partial x^k}{\partial y^\alpha}=S^k_{ij}\frac{\partial x^i}{\partial y^\beta}\frac{\partial x^j}{\partial y^\gamma},$$

this equation means that torsion of a hypersurface is generated by torsion embedding space Y^n.

From the last identity, we can see that our definition of Y^{n-1} torsion tensor is correct and coherent.

Below we use the mixed tensors values enumerated two types of indices, while Latin indices refer to the containing space Y^n and responsive to the coordinate transformation x^i, and Greek indices belong to the space hypersurface $Y^{n-\mathbf{1}}$ and responsive to the transformation of coordinate y^α. The index i is not responsive to the coordinate y^α transformation into $Y^{n-\mathbf{1}}$, and the index α does not respond to the coordinate x^i transformation in Y^n. For example, the formula to calculate the covariant derivative of a mixed tensor:

$$A^{i\alpha}_{j\beta;\gamma}=A^{i\alpha}_{j\beta,\gamma}+\Gamma^i_{pk}A^{p\alpha}_{j\beta}\frac{\partial x^k}{\partial y^\gamma}-\Gamma^p_{jq}A^{i\alpha}_{p\beta}\frac{\partial x^q}{\partial y^\gamma}+G^\alpha_{\eta\gamma}A^{i\eta}_{j\beta}-G^\eta_{\beta\gamma}A^{i\alpha}_{j\eta}, \tag{7}$$

and similarly, we define differential of mixed tensor

$$DA^{i\alpha}_{j\beta}=dA^{i\alpha}_{j\beta}+\Gamma^i_{pk}A^{p\alpha}_{j\beta}dx^k-\Gamma^p_{jk}A^{i\alpha}_{p\beta}dx^k+G^\alpha_{\eta\gamma}A^{i\eta}_{j\beta}du^\gamma-G^\eta_{\beta\gamma}A^{i\alpha}_{j\eta}du^\gamma \tag{8}$$

The direct calculations lead us to formulas:

$$u_{i;\alpha;\beta}-u_{i;\beta;\alpha}=R^p_{kli}u_p\frac{\partial x^l}{\partial y^\alpha}\frac{\partial x^k}{\partial y^\beta}+S^q_{kl}u_{i;q}\frac{\partial x^l}{\partial y^\alpha}\frac{\partial x^k}{\partial y^\beta}, \tag{9}$$

$$u_{\gamma;\alpha;\beta}-u_{\gamma;\beta;\alpha}=R^\eta_{\beta\alpha\gamma}u_\eta+T^\eta_{\beta\alpha}u_{\gamma;\eta}, \tag{10}$$

where $R^\eta_{\beta\alpha\gamma}$ - curvature tensor of space $Y^{n-\mathbf{1}}$ compiled by using the components of connection $G^\alpha_{\beta\gamma}$ of hypersurface $Y^{n-\mathbf{1}}$.

Let us considered the system of values

$$\xi^i_\alpha=\frac{\partial x^i}{\partial y^\alpha}. \tag{11}$$

At each point of the hypersurface $Y^{n-\mathbf{1}}$ we can build rapper consisting of the vectors:

$$\xi^i_{\mathbf{1}},\ldots,\xi^i_{n-\mathbf{1}},\nu^i, \tag{12}$$

where $\xi^i_{\mathbf{1}},\ldots,\xi^i_{n-\mathbf{1}}$ linearly independent tangent vectors and ν^i normal vector, defined as the metric and connection agreed.

Next, we act formally, the idea is the same as in the classical case, and we will indicate significant new moments. We compute the derivative of the mixed tensors ξ^i_α

$$\xi^i_{\alpha;\gamma} = \xi^i_{\alpha,\gamma} + \Gamma^i_{pq}\xi^p_\alpha \frac{\partial x^q}{\partial y^\gamma} - G^\eta_{\alpha\gamma}\xi^i_\eta . \quad (13)$$

In contrast to the case of torsion-free connection, we have the equality

$$\xi^i_{\alpha;\gamma} - \xi^i_{\gamma;\alpha} = S^i_{pq}\xi^p_\alpha\xi^q_\gamma + T^\eta_{\gamma\alpha}\xi^i_\eta , \quad (14)$$

but, we have

$$\xi^i_{\alpha;\gamma} - \xi^i_{\gamma;\alpha} = \left(S^i_{pq} + S^k_{qp}\xi^i_\eta\xi^m_\mu g_{km}a^{\mu\eta}\right)\xi^p_\alpha\xi^q_\gamma = \mathbf{0} . \quad (15)$$

Next, we permute the indices in the equation:

$$\mathbf{0} = a_{\alpha\beta;\gamma} = \left(g_{ij}\xi^i_\alpha\xi^j_\beta\right)_{;\gamma} = g_{ij}\xi^i_{\alpha;\gamma}\xi^j_\beta + g_{ij}\xi^i_\alpha\xi^j_{\beta;\gamma} , \quad (16)$$

we obtain

$$g_{ij}\xi^i_{\alpha;\gamma}\xi^j_\beta = \mathbf{0} .$$

Hence, we can write decomposition:

$$\xi^i_{\beta;\alpha} = \pi_{\alpha\beta}\nu^i . \quad (17)$$

Remark 1. Set $\pi_{\alpha\beta}$ is a tensor, which similar to the second fundamental tensor of hypersurfaces Y^{n-1}, but its structure in this space substantially different from the case of Riemannian spaces with zero torsion.

We remark that equation $S^i_{pq}\xi^p_\beta\xi^q_\alpha = (\xi^i_{\beta;\alpha} - \xi^i_{\alpha;\beta}) + T^\eta_{\beta\alpha}\xi^i_\eta$ is the simple result of definition.

So we have equality

$$\pi_{\alpha\beta} - \pi_{\beta\alpha} = g_{ij}S^i_{pq}\xi^p_\beta\xi^q_\alpha\nu^j .$$

Then we have obtained by differentiating $g_{ij}\nu^i\xi^j_\alpha = \mathbf{0}$ by γ:

$$g_{ij}\nu^i_{;\gamma}\xi^i_\alpha = -\pi_{\gamma\alpha} . \quad (18)$$

Similarly, by differentiating $g_{ij}\nu^i\nu^j = \mathbf{1}$ by γ, we obtain:

$$\nu^i_{;\gamma} = -a^{\eta\mu}\pi_{\mu\gamma}\xi^i_\eta . \quad (19)$$

From $\pi_{\alpha\beta} - \pi_{\beta\alpha} = g_{ij} S^{i}_{pq} \xi^{p}_{\beta} \xi^{q}_{\alpha} \nu^{j}$ and $T^{\alpha}_{\beta\gamma} \xi^{k}_{\alpha} = S^{k}_{ij} \xi^{i}_{\beta} \xi^{j}_{\gamma}$ (when embedding is smooth enough) one can obtain that $\pi_{\alpha\beta}$ has to be symmetrical. So, asymmetry of a tensor $\pi_{\alpha\beta}$ is induced by embedding.
Formula (17) and (19) characterize the change of vectors in the small accompanying frame relative to this frame itself.

We associate with Y^{n-1} a coordinate system in Y^{n}, which can be denoted $u^{1},...,u^{n-1},u^{n}$, by the rule

$$u^{1} = y^{1},...,u^{n-1} = y^{n-1}, u^{n} = z,$$

with a new metric $\tilde{g}_{ik}$ defined by $\tilde{g}_{\alpha\beta} = a_{\alpha\beta}$, $\tilde{g}_{n\alpha} = 0$, $\tilde{g}_{nn} = 1$. Where z is a geodesic line directed along ν^{i} - the normal to hypersurface?

Since the rank of the matrix $\left[\frac{\partial x^{i}}{\partial y^{\alpha}}\right]$ equal $n-1$ suppose that $rank\left[\frac{\partial x^{\gamma}}{\partial y^{\alpha}}\right] > 0$ then exist the solution of a system of equations

$$x^{1} = x^{1}\left(y^{1},...,y^{n-1}\right),$$
$$.....,$$
$$x^{n-1} = x^{n-1}\left(y^{1},...,y^{n-1}\right),$$

which we denote by

$$u^{1} = y^{1} = y^{1}(x^{1},...,x^{n-1}),$$
$$.....,$$
$$u^{n-1} = y^{n-1} = y^{n-1}(x^{1},...,x^{n-1}),$$

and

$$u^{n} = z = z(x^{1},...,x^{n-1},x^{n})$$

herewith metric tensor $\tilde{g}^{ik}$ equals

$$\tilde{g}^{ik} = \xi^{i}_{\alpha} \xi^{k}_{\beta} a^{\alpha\beta} + \nu^{i} \nu^{k}.$$

If we consider the system (17) and (19) from a geometric point of view, then we can formulate the problem for differential equations, where the unknowns are considered the functions and are given (known) g_{ik}, $a_{\alpha\beta}$, S^{i}_{jk}, $T^{\gamma}_{\alpha\beta}$, $\pi_{\alpha\beta}$, as a function of $y^{\mathbf{1}},..., y^{n-\mathbf{1}}$. Then the connection coefficients Γ^{i}_{jk} as a function of $y^{\mathbf{1}},..., y^{n-\mathbf{1}}$ must be considered as a known, and it means that we know exactly how the hypersurface $Y^{n-\mathbf{1}}$ is embedded in the space Y^{n}, but then ξ^{i}_{α}, ν^{i} we have to consider as the known functions of $y^{\mathbf{1}},..., y^{n-\mathbf{1}}$ and so the problem makes no sense.

Further, we obtain:

$$\xi^i_{\beta;\chi;\lambda} - \xi^i_{\beta;\lambda;\chi} = -R^i_{klp}\xi^k_\lambda \xi^l_\chi \xi^p_\beta + R^\sigma_{\lambda\chi\beta}\xi^i_\sigma + T^\sigma_{\lambda\chi}\xi^i_{\beta;\sigma} =$$
$$= \left(\pi_{\chi\beta;\lambda} - \pi_{\lambda\beta;\chi}\right)\nu^i - \left(\pi_{\chi\beta}\pi_{\eta\lambda}a^{\eta\sigma} - \pi_{\lambda\beta}\pi_{\eta\chi}a^{\eta\sigma}\right)\xi^i_\sigma. \tag{20}$$

Equation (3.8) is multiplying by $g_{ij}\xi^j_\alpha$, we have:

$$R_{\alpha\lambda\chi\beta} = R_{iklp}\xi^k_\lambda \xi^l_\chi \xi^p_\beta \xi^i_\alpha - \left(\pi_{\chi\beta}\pi_{\alpha\lambda} - \pi_{\lambda\beta}\pi_{\alpha\chi}\right). \tag{21}$$

Similarly, we derive a formula:

$$\nu^i_{;\chi;\lambda} - \nu^i_{;\lambda;\chi} = -R^i_{klp}\xi^k_\lambda \xi^l_\chi \nu^p + T^\sigma_{\lambda\chi}\nu^i_{;\sigma} =$$
$$= \left(\pi_{\eta\lambda;\chi}a^{\eta\sigma} - \pi_{\eta\chi;\lambda}a^{\eta\sigma}\right)\xi^i_\sigma. \tag{22}$$

We contract (20) with $g_{ij}\nu^j$, then:

$$-R_{iklp}\xi^k_\lambda \xi^l_\chi \xi^p_\beta \nu^i + T^\sigma_{\lambda\chi}\pi_{\sigma\beta} = \pi_{\chi\beta;\lambda} - \pi_{\lambda\beta;\chi}. \tag{23}$$

Formula (22) is multiplying by $g_{ij}\xi^j_\alpha$, we concluded that:

$$-R_{iklp}\xi^k_\lambda \xi^l_\chi \nu^p \xi^i_\alpha + T^\sigma_{\lambda\chi}\pi_{\alpha\sigma} = \pi_{\alpha\lambda;\chi} - \pi_{\alpha\chi;\lambda}. \tag{24}$$

Remark 2. If (22) contract with $g_{ij}\nu^j$, then we obtain identically zero.

Thus, we have the two types of formulas. Formula (21) does not contain the torsion tensor explicitly, but it is counted in the tensor $\pi_{\alpha\beta}$.

In the formula (22) the torsion tensor of the hypersurface present explicitly and in the form of coefficients of $\pi_{\alpha\beta}$*, and appears in the calculation of the covariant derivative.*

Now, let us denoted $\vartheta_{\alpha\beta}$ symmetrical tensor $g_{ij}\nu^i_{;\alpha}\nu^j_{;\beta}$ and we have

$$\vartheta_{\alpha\beta} = g_{ij}\nu^i_{;\alpha}\nu^j_{;\beta} = g_{ij}a^{\eta\mu}\pi_{\mu\alpha}\xi^i_\eta a^{\chi\delta}\pi_{\delta\beta}\xi^j_\chi = g_{ij}\pi^\eta_\alpha \pi^\chi_\beta \xi^i_\eta \xi^j_\chi = a_{\eta\chi}\pi^\eta_\alpha \pi^\chi_\beta = a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta},$$

or

$$\vartheta_{\alpha\beta} = g_{ij}\nu^i_{;\alpha}\nu^j_{;\beta} = g_{ij}\left(\nu^i_{,\alpha} + \Gamma^i_{lk}\nu^k \xi^l_\alpha\right)\left(\nu^j_{,\beta} + \Gamma^j_{pq}\nu^q \xi^p_\beta\right) =$$
$$= g_{ij}\nu^i_{,\alpha}\nu^j_{,\beta} + g_{ij}\nu^j_{,\beta}\Gamma^i_{lk}\nu^k \xi^l_\alpha + g_{ij}\nu^i_{,\alpha}\Gamma^j_{pq}\nu^q \xi^p_\beta + g_{ij}\Gamma^i_{lk}\Gamma^j_{pq}\nu^k \xi^l_\alpha \nu^q \xi^p_\beta \tag{25}$$

therefore, we see that asymmetrical part vanished.

We denote $M = \frac{1}{2} a^{\alpha\beta} \pi_{\alpha\beta}$, we obtain

$$\vartheta_{\alpha\beta} = a_{\eta\chi} \pi_{\alpha}^{\eta} \pi_{\beta}^{\chi} = a^{\eta\chi} \pi_{\eta\alpha} \pi_{\chi\beta}, \qquad \mathbf{2} M a^{\eta\chi} = a^{\alpha\beta} \pi_{\alpha\beta} a^{\eta\chi} \tag{26}$$

Then

$$a^{\eta\chi} \left(\pi_{\eta\alpha} \pi_{\chi\beta} - a^{\alpha\beta} \pi_{\alpha\beta} \pi_{\eta\chi} \right) = -\frac{\pi}{a} a_{\alpha\beta} \tag{27}$$

and

$$a^{\eta\chi} \left(\pi_{\eta\alpha} \pi_{\chi\beta} - a^{\alpha\beta} \pi_{\alpha\beta} \pi_{\eta\chi} \right) + \frac{\pi}{a} a_{\alpha\beta} = \mathbf{0},$$

next, we can write

$$\vartheta_{\alpha\beta} - \mathbf{2} M \pi_{\alpha\beta} + \frac{\pi}{a} a_{\alpha\beta} = \mathbf{0}$$

and we obtain

$$\vartheta_{\alpha\beta} = \mathbf{2} M \pi_{\alpha\beta} - \frac{\pi}{a} a_{\alpha\beta}.$$

We calculate

$$\begin{aligned} \vartheta_{\alpha\beta;\omega} - \vartheta_{\omega\beta;\alpha} &= a^{\eta\chi} \pi_{\eta\alpha;\omega} \pi_{\chi\beta} + a^{\eta\chi} \pi_{\eta\alpha} \pi_{\chi\beta;\omega} - a^{\eta\chi} \pi_{\eta\omega;\alpha} \pi_{\chi\beta} - a^{\eta\chi} \pi_{\eta\omega} \pi_{\chi\beta;\alpha} = \\ &= a^{\eta\chi} \pi_{\chi\beta} \left(\pi_{\eta\alpha;\omega} - \pi_{\eta\omega;\alpha} \right) + a^{\eta\chi} \pi_{\eta\alpha} \pi_{\chi\beta;\omega} - a^{\eta\chi} \pi_{\eta\omega} \pi_{\chi\beta;\alpha} \end{aligned}$$

and

$$\begin{aligned} \vartheta_{\alpha\beta;\omega} - \vartheta_{\omega\beta;\alpha} &= a^{\eta\chi} \pi_{\chi\beta} \left(-R_{iklp} \xi_{\eta}^{i} \xi_{\alpha}^{k} \xi_{\omega}^{l} \nu^{p} + T_{\omega\alpha}^{\sigma} \pi_{\eta\sigma} \right) + a^{\eta\chi} \pi_{\eta\alpha} \pi_{\chi\beta;\omega} - a^{\eta\chi} \pi_{\eta\omega} \pi_{\chi\beta;\alpha} = \\ &= -R_{iklp} \xi_{\eta}^{i} \xi_{\alpha}^{k} \xi_{\omega}^{l} \nu^{p} a^{\eta\chi} \pi_{\chi\beta} + T_{\omega\alpha}^{\sigma} \pi_{\eta\sigma} a^{\eta\chi} \pi_{\chi\beta} + a^{\eta\chi} \pi_{\eta\alpha} \pi_{\chi\beta;\omega} - a^{\eta\chi} \pi_{\eta\omega} \pi_{\chi\beta;\alpha}. \end{aligned} \tag{28}$$

A tensor can be associated with the square of the angle between normal and adjacent normal $\vartheta_{\alpha\beta} dy^{\alpha} dy^{\beta} = d\varphi^{\mathbf{2}}$. *So, let in space* Y^n *with coordinates* $x^1,...,x^n$ *given the system of non-degenerate equations* $x^i = x^i \left(y^1,..., y^{n-1} \right)$ *so is determined the hypersurface* Y^{n-1} *and the metric and torsion of* Y^{n-1} *and since the connection of* Y^{n-1}. *We can consider the hypersurface like* Y^{n-1} *space and so we obtain all internal (intrinsic) geometry structure of* Y^{n-1}, *but formulas* $x^i = x^i \left(y^1,..., y^{n-1} \right)$ *define more, then internal (intrinsic) geometry structure of* Y^{n-1}, *they define external geometry of* Y^{n-1} *(imbedding) as well. External geometry or "how the hypersurface* Y^{n-1} *is imbedded" define by one of the tensors* $\pi_{\alpha\beta}$ *or* $\vartheta_{\alpha\beta}$

which determinate position of a hypersurface in Y^n *space. As an example, internal (intrinsic) geometry in* Y^{n-1} *we considered geodesic on* Y^{n-1}.

3 Geodesic on Hypersurface

According to definition geodesic on Y^{n-1} determined by a formula

$$\frac{d^2 y^\alpha}{ds^2} = -G^\alpha_{\beta\gamma}\frac{dy^\beta}{ds}\frac{dy^\gamma}{ds}. \tag{29}$$

Let a curve: $y^\alpha = y^\alpha(\tau), \quad \tau \in [\tau_1; \tau_2]$ We calculate the variation of the length of geodesic δS of the curve S:

$$\delta\left(a_{\alpha\beta}\frac{dy^\alpha}{d\tau}\frac{dy^\beta}{d\tau}\right) = a_{\alpha\beta}\tilde{D}\frac{dy^\alpha}{d\tau}\frac{dy^\beta}{d\tau} + a_{\alpha\beta}\frac{dy^\alpha}{d\tau}\tilde{D}\frac{dy^\beta}{d\tau} = 2a_{\alpha\beta}\frac{dy^\alpha}{d\tau}\tilde{D}\frac{dy^\beta}{d\tau}$$

$$\tilde{D}\frac{dy^\alpha}{d\tau} = \delta\frac{dy^\alpha}{d\tau} + G^\alpha_{\beta\gamma}\frac{dy^\beta}{d\tau}\delta y^\gamma$$

$$D\frac{\delta y^\alpha}{d\tau} = \frac{d}{d\tau}\delta y^\alpha + G^\alpha_{\gamma\beta}\frac{dy^\gamma}{d\tau}\delta y^\beta$$

$$\tilde{D}\frac{\delta y^\alpha}{d\tau} = D\frac{dy^\alpha}{d\tau} + T^\alpha_{\beta\gamma}\frac{dy^\beta}{d\tau}\delta y^\gamma$$

where denotes $\tilde{D}$ the absolute differential at the parameter curves of the family at a constant value τ, and D is absolute differential displacement $d\tau$ curve at a constant parameter of the family, then

$$\delta\left(g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right) = 2g_{ij}\frac{dx^i}{dt}\left(D\frac{\delta x^j}{dt} + S^j_{pk}\frac{dx^p}{dt}\delta x^k\right),$$

$$\delta s = \int_{\tau_1}^{\tau_2} a_{\alpha\beta}\frac{dy^\alpha}{ds}D\delta y^\beta + \int_{\tau_1}^{\tau_2} a_{\alpha\beta}T^\beta_{\gamma\lambda}\frac{dy^\alpha}{d\tau}dy^\gamma\delta y^\lambda =$$

$$\int_{\tau_1}^{\tau_2} D\left(a_{\alpha\beta}\frac{dy^\alpha}{ds}\delta y^\beta\right) - \int_{\tau_1}^{\tau_2} a_{\alpha\beta}D\frac{dy^\alpha}{ds}\delta y^\beta + \int_{\tau_1}^{\tau_2} a_{\alpha\beta}T^\beta_{\gamma\lambda}\frac{dy^\alpha}{d\tau}dy^\gamma\delta y^\lambda$$

since the ends of the variable curve are fixed

$$\delta s = \int_{\tau_1}^{\tau_2}\left(a_{\alpha\beta}T^\beta_{\gamma\lambda}\frac{dy^\alpha}{d\tau}dy^\gamma\delta y^\lambda - a_{\alpha\beta}D\frac{dy^\alpha}{ds}\delta y^\beta\right),$$

suppose considered curve has a fixed length (analytically $\delta s = 0$), then we obtain:

$$\delta s = \int_{\tau_1}^{\tau_2} \left(a_{\alpha\beta} T^{\beta}_{\gamma\lambda} \frac{dy^{\alpha}}{d\tau} dy^{\gamma} \delta y^{\lambda} - a_{\alpha\beta} D \frac{dy^{\alpha}}{ds} \delta y^{\beta} \right) = 0.$$

By the fundamental lemma of calculus of variations, it follows:

$$a_{\alpha\lambda} T^{\lambda}_{\gamma\beta} \frac{dy^{\alpha}}{d\tau} dy^{\gamma} - a_{\alpha\beta} D \frac{dy^{\alpha}}{ds} = 0.$$

The variation of the length of the geodesic is:

$$\delta s = \int_{t_1}^{t_2} a_{\alpha\lambda} T^{\lambda}_{\gamma\beta} \frac{dy^{\alpha}}{d\tau} dy^{\gamma}. \tag{30}$$

We remark, that the geodesics on Y^{n-1} which are determined by connection $G^{\alpha}_{\beta\gamma}$ don't depend on terms that contain tensor $T^{\alpha}_{\beta\gamma}$.

Now, we can construct a semi-geodesics coordinate system in any point of Y^{n-1}, but we can't integrate it.

So, similarly to embedding space Y^n, we define the geodesic lines in Y^{n-1} space by formula

$$\frac{d^2 y^{\alpha}}{ds^2} + G^{\alpha}_{\beta\gamma} \frac{dy^{\beta}}{ds} \frac{dy^{\gamma}}{ds} = 0,$$

and a variation of the length of the geodetic δs on Y^{n-1}

$$\delta s = \int_{t_1}^{t_2} a_{\alpha\beta} T^{\beta}_{\gamma\eta} \frac{dy^{\alpha}}{dt} dy^{\gamma} \delta y^{\eta}, \tag{31}$$

which depends on torsion of the hypersurface Y^{n-1} and can be express in terms of torsion in Y^n.

We define the geodesic hypersurface as the hypersurface Y^{n-1} on which any geodesic lines in Y^{n-1} is a geodesic line in embedding space Y^n.

4 Example of Geodesics in Y^{n-1} Space with Euclidean Metric

Let us studied an example, of a geodesic on hypersurface Y^3 embedding in the four-dimensional Y^4 space with diagonal Euclidean metric. We assume that metric is

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and torsion is

$$S_{ik}^{1}=\begin{pmatrix}0&0&0&0\\0&0&a&0\\0&-a&0&0\\0&0&0&0\end{pmatrix},\quad S_{ik}^{2}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad S_{ik}^{3}=\begin{pmatrix}0&a&0&0\\-a&0&a&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad S_{ik}^{4}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \qquad (32)$$

Let us computed the connection with this metric, in general case, we have

$$\Gamma_{ik}^{1}=\begin{pmatrix}0&0&0&0\\S_{21}^{1}&S_{21}^{2}&\frac{1}{2}\left(S_{23}^{1}+S_{31}^{2}+S_{21}^{3}\right)&\frac{1}{2}\left(S_{24}^{1}+S_{41}^{2}+S_{21}^{4}\right)\\S_{31}^{1}&\frac{1}{2}\left(S_{32}^{1}+S_{31}^{2}+S_{21}^{3}\right)&S_{31}^{3}&\frac{1}{2}\left(S_{34}^{1}+S_{41}^{3}+S_{31}^{4}\right)\\S_{41}^{1}&\frac{1}{2}\left(S_{42}^{1}+S_{41}^{2}+S_{21}^{4}\right)&\frac{1}{2}\left(S_{43}^{1}+S_{41}^{3}+S_{31}^{4}\right)&S_{41}^{4}\end{pmatrix},$$

$$\Gamma_{ik}^{2}=\begin{pmatrix}S_{12}^{1}&S_{12}^{2}&\frac{1}{2}\left(S_{32}^{1}+S_{13}^{2}+S_{12}^{3}\right)&\frac{1}{2}\left(S_{42}^{1}+S_{14}^{2}+S_{12}^{4}\right)\\0&0&0&0\\\frac{1}{2}\left(S_{32}^{1}+S_{31}^{2}+S_{12}^{3}\right)&S_{32}^{2}&S_{32}^{3}&\frac{1}{2}\left(S_{43}^{2}+S_{42}^{3}+S_{32}^{4}\right)\\\frac{1}{2}\left(S_{42}^{1}+S_{43}^{2}+S_{12}^{3}\right)&S_{42}^{2}&\frac{1}{2}\left(S_{34}^{2}+S_{42}^{3}+S_{32}^{4}\right)&S_{42}^{4}\end{pmatrix},$$

$$\Gamma_{ik}^{3}=\begin{pmatrix}S_{13}^{1}&\frac{1}{2}\left(S_{23}^{1}+S_{13}^{2}+S_{12}^{3}\right)&S_{13}^{3}&\frac{1}{2}\left(S_{43}^{1}+S_{14}^{3}+S_{13}^{4}\right)\\\frac{1}{2}\left(S_{23}^{1}+S_{13}^{2}+S_{21}^{3}\right)&S_{23}^{2}&S_{23}^{3}&\frac{1}{2}\left(S_{43}^{2}+S_{24}^{3}+S_{23}^{4}\right)\\0&0&0&0\\\frac{1}{2}\left(S_{43}^{1}+S_{13}^{3}+S_{13}^{4}\right)&\frac{1}{2}\left(S_{43}^{2}+S_{42}^{3}+S_{23}^{4}\right)&S_{43}^{3}&S_{43}^{4}\end{pmatrix},$$

$$\Gamma_{ik}^{4}=\begin{pmatrix}S_{14}^{1}&\frac{1}{2}\left(S_{24}^{1}+S_{14}^{2}+S_{12}^{4}\right)&\frac{1}{2}\left(S_{34}^{1}+S_{14}^{3}+S_{13}^{4}\right)&S_{14}^{4}\\\frac{1}{2}\left(S_{24}^{1}+S_{14}^{2}+S_{21}^{4}\right)&S_{24}^{2}&\frac{1}{2}\left(S_{34}^{2}+S_{24}^{3}+S_{23}^{4}\right)&S_{24}^{4}\\\frac{1}{2}\left(S_{34}^{1}+S_{14}^{3}+S_{31}^{4}\right)&\frac{1}{2}\left(S_{34}^{2}+S_{24}^{3}+S_{32}^{4}\right)&S_{34}^{3}&S_{34}^{4}\\0&0&0&0\end{pmatrix}$$

and, we obtain

$$\Gamma^1_{ik}=\begin{pmatrix}0&0&0&0\\-a&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \Gamma^2_{ik}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \Gamma^3_{ik}=\begin{pmatrix}0&a&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \Gamma^4_{ik}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \qquad (33)$$

Now, let us denoted the coordinates of hypersurface by u^1, u^2, u^3, and we assume that hypersurface is given by formulas

$$x^1=u^1,$$
$$x^2=u^2,$$
$$x^3=u^3,$$
$$x^4=0;$$

and for hypersurface, we have

$$T^1_{ik}=\begin{pmatrix}0&0&0\\0&0&a\\0&-a&0\end{pmatrix},\quad T^2_{ik}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad T^3_{ik}=\begin{pmatrix}0&a&0\\-a&0&0\\0&0&0\end{pmatrix}. \qquad (34)$$

We calculate connection of hypersurface

$$G^1_{ik}=\begin{pmatrix}0&0&0\\-a&0&0\\0&0&0\end{pmatrix},\quad G^2_{ik}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad G^3_{ik}=\begin{pmatrix}0&a&0\\0&0&0\\0&0&0\end{pmatrix}; \qquad (35)$$

the geodesics determined by a formula

$$\frac{d^2u^\alpha}{ds^2}+G^\alpha_{\beta\eta}\frac{dx^\beta}{ds}\frac{dx^\eta}{ds}=0$$

and we can write formulas

$$\begin{cases}\dfrac{d^2u^1}{ds^2}-a\dfrac{du^2}{ds}\dfrac{du^3}{ds}=0\\ \dfrac{d^2u^2}{ds^2}=0\\ \dfrac{d^2u^3}{ds^2}+a\dfrac{du^1}{ds}\dfrac{du^2}{ds}=0.\end{cases} \qquad (36)$$

This system that describes geodesics has general and particular solutions. The general solution of this system is

$$u^1 = -\frac{k_1}{k_2 a}\cos\left(k_2 as\right) - \frac{k_3}{k_2 a}\sin\left(k_2 as\right) + k_4$$
$$u^2 = k_2 s + k_5$$
$$u^3 = \frac{k_1}{k_2 a}\sin\left(k_2 as\right) - \frac{k_3}{k_2 a}\cos\left(k_2 as\right) + k_6 \qquad (37)$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ are independent constants?

The particular solution can be written in the form

$$u^1 = M_1 s + M_2$$
$$u^2 = M_3$$
$$u^3 = M_4 s + M_5 \qquad (38)$$

where M_1, M_2, M_3, M_4, M_5 are arbitrary constants?

5 Conclusions and Recommendations

We have studied the geometrical structure of the hypersurface Y^{n-1}, the space that is generated jointly and agreed by the metric tensor and the torsion tensor. We have presented the structure of the curvature tensor and studied its special features and for this tensor obtained analogue Ricci - Jacobi identity. The geodesic lines equation has been researched. We have shown that the structure of tensor $\pi_{\alpha\beta}$, which is similar to the second fundamental tensor of hypersurfaces $Y^{n-\mathbf{1}}$, is substantially different from the case of Riemannian spaces with zero torsion. Then we have obtained formulas which characterize the change of vectors in accompanying basis relative to this basis itself in the small. The geodesic in Y^{n-1} with metric and torsion in case of Euclidean metric in the three-dimensional space.

Competing Interests

Author has declared that no competing interests exist.

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