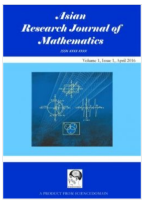

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On the Co-Common Neighborhood Domination Number

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Authors' contributions

This work was carried out in collaboration between all authors. All authors contributed equally and significantly to writing this paper. All authors read, developed and approved the final manuscript.

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Abstract

In this paper, we introduce the concept of co-common neighborhood domination number (CCN-domination number) $\gamma_{ccn}(G)$ of a graph G and we study its relation with the standard domination number $\gamma(G)$. We also define CCN-independence number $\beta_{ccn}(G)$, total CCN-domination number $\gamma_{tccn}(G)$, CCN-covering number $\alpha_{ccn}(G)$ and CCN-domatic number.

Keywords: Co-Common neighborhood domination; Co-Common neighborhood dominating set.

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1 Introduction

In the last 60 years, Graph theory has seen an explosive growth due to interaction with areas like computer science, electrical and communication engineering, Operations Research etc. Perhaps the

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fastest growing area within graph theory is the study of domination, the reason being its many and varied applications in such fields as social sciences, communication networks, algorithm designs, computational complexity etc. There are several types of domination depending upon the nature of domination and the nature of the dominating set [1].

Alwardi and Soner [2],[3], introduced and studied the concept of common neighborhood domination which motivated us to introduce the concept co-common neighborhood domination. All the graph considered here are finite and undirected with no loops and multiple edges. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G, respectively. we use $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v respectively. Degree of a vertex v is denoted by $deg(v)$ where $deg(v) = |N(v)|$, the maximum and the minimum degree of a graph G is denoted by $\Delta(G), \delta(G)$ respectively. The distance $d(u, v)$ from a vertex u to a vertex v in a connected graph G is the minimum of the lengths of the $u - v$ paths in G. The eccentricity $ec(v)$ of v is $\max_{u\in V} d(u, v)$. The radius $rad(G)$ of G is $\min_{v\in V} ec(v)$. The diameter $diam(G)$ of G is $\max_{u\in V} ec(v)$.

A set D of vertices in a graph G is a dominating set if every vertex in $V-D$ is adjacent to some vertex in D . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G . A set S of vertices in a graph G is a vertex covering or a vertex cover of G if every edge in G is incident on at least one vertex in S. For more details about parameters of domination number, we refer to [1], [4], [5], [6], [7], [8], [9] . A strongly regular graph with parameters (n, k, λ, μ) is k -regular graph with n vertices such that for any two adjacent vertices have λ common neighbors, and any two non- adjacent vertices have μ common neighbors. The join $G+H$ of two graphs G and H is the graph with vertex set $V(G+H) = V(G)\cup V(H)$ and edge set $E(G+H) = E(G)\cup E(H)\cup \{uv : u \in V(G), v \in V(H)\}$. The corona $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.

2 CNN-Dominating Sets

Definition 2.1. Let $G = (V, E)$ be simple graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. For $i \neq j$, the co-common neighborhood of the vertices v_i and v_j, denoted by $\Gamma^{'}(v_i, v_j)$, is the set of vertices, different from v_i and v_j , which are not adjacent to both v_i and v_j [10].

Definition 2.2. A subset S of V is called co-common neighborhood dominating set (CCN-dominating set) if for every vertex $v \in V - S$, there exists a vertex $u \in S$ such that $uv \in E(G)$ and $\left|\Gamma^{'}(v_i, v_j)\right| \geq 1$. The minimum cardinality of a co-common neighborhood dominating set denoted by γ_{ccn} and is called co-common neighborhood domination number (CCN-domination number) of G. It is clear that CCN-domination number is defined for any graph.

Proposition 2.1.

(i) *For any complete graph* K_n, $\gamma_{ccn}(K_n) = n$.

(ii) *For any path* P_n, *where* $n > 3$, $\gamma_{ccn}(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

(iii) *For any cycle* C_n, *where* $n > 4$, $\gamma_{ccn}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

(iv) *For any complete bipartite graph* $K_{n,m}$, $\gamma_{ccn}(K_{n,m}) = n + m$.

(v) *For any wheel graph with* $n > 5$ *vertices, ,* $\gamma_{ccn}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil + 1$.

Definition 2.3. A co-common neighborhood dominating set S is said to be minimal co-common neighborhood dominating set if no proper subset of S is co-common neighborhood dominating set.

Definition 2.4. A minimal co-common neighborhood dominating set S of maximum cardinality is called Γ_{ccn}-set and its cardinality is denoted by $\Gamma_{ccn}(G)$..

Let $G = (V, E)$ be a graph and $u \in V$ be such that $uv \notin E(G)$ or $\left| \Gamma^{'}(u, v) \right| = 0$ for all $v \in V$. Then u is in every co-common neighborhood dominating set, such points are called co-common neighborhood isolated vertices. Let I_{ccn} denote the set of all co-common neighborhood isolated vertices of G. Hence $I \subseteq I_{ccn} \subseteq S$, where I is the set of isolated vertices and S is the minimum CCN-dominating set of G.

Definition 2.5. Let $G = (V, E)$ be a graph. For any vertex $u \in V$ the CCN-neighborhood of u denoted by $N_{ccn}(u)$ is defined as $N_{ccn}(u) = \{v \in V : uv \in E(G) \text{ and } \left| \Gamma^{'}(u, v) \right| \geq 1\}$. The cardinality of $N_{ccn}(u)$ is called the co-common neighborhood degree (CCN-degree) of u and denoted by $deg_{ccn}(u)$ in G, and $N_{ccn}[u] = N_{ccn}(u) \cup \{u\}$. The maximum and minimum co-common neighborhood degree of a vertex in G are denoted respectively by $\Delta_{ccn}(G)$ and $\delta_{ccn}(G)$. That is $\Delta_{ccn}(G) = \max_{u \in V} |N_{ccn}(u)|$ and $\delta_{ccn}(G) = \min_{u \in V} |N_{ccn}(u)|$.

Definition 2.6. If u and v are any two vertices in V such that $uv \in E(G)$ and $\left| \Gamma^{'}(u, v) \right| \geq 1$, then we say u is co-common neighborhood adjacent (CCN-adjacent) to v, or u is CCN-dominate v.

Example 2.1. *Let G be a graph as in Fig. 1. Then we have:* $N_{ccn}(1) = \phi$, $deg_{ccn}(1) = 0$, $deg(1) = 5$, $N_{ccn}(2) = \{3, 6\}$, $deg_{ccn}(2) = 2$, $deg(2) = 3$, $N_{ccn}(3) = \{2, 4\}$, $deg_{ccn}(3) = 2$, $\text{deg}(3) = 3$, $N_{ccn}(4) = \{3, 5\}$, $deg_{ccn}(4) = 2$, $\text{deg}(4) = 3$, $N_{ccn}(5) = \{4, 6\}$, $deg_{ccn}(5) = 2$, $deg(5) = 3$, *and* $N_{ccn}(6) = \{1, 5\}$, $deg_{ccn}(6) = 2$, $deg(6) = 3$.

Also $\{1\}$ *is minimum dominating set and* $\{1, 2, 4\}$ *is minimum CCN-dominating set, so* $\gamma_{ccn}(G) = 3$, *but* $\gamma(G) = 1$. *The vertices* 1 *is CCN-isolated vertex but not isolated vertex.*

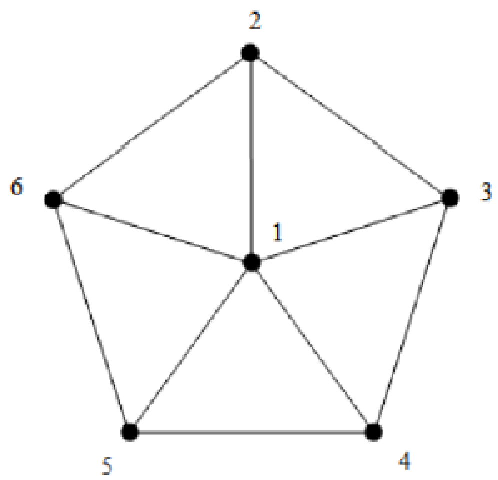

Fig. 1. Wheel graph W_6

Theorem 2.2. *A co-common neighborhood dominating set S is minimal if and only if for every vertex* $u \in S$ *one of the following holds.*

(i) u is not CCN–adjacent to any vertex in S.
(ii) There exists a vertex $v \in V - S$ *such that* $N_{ccn}(v) \cap S = \{u\}$.

Proof. Suppose that S is minimal co-common neighborhood dominating set and suppose that $u \in S$. Then $S - \{u\}$ is not CCN-dominating set, so there exists a vertex $v \in (V - S) \cup \{u\}$ which is not CCN-adjacent to any vertex in $S - \{u\}$.

Case1: $v = u$, then u is not CCN-adjacent to any vertex in S.
Case2: $v \neq u$, then $v \in V - S$ and not CCN-adjacent to any vertex in $S - \{u\}$, but is CCN-dominated by S, then v is CCN-adjacent only to vertex u in S. That means$N_D(v) \cap S = \{u\}$.

Conversely, suppose that S is CCN-dominating set and for each vertex $u \in S$ one of the two condition holds. We want to prove that S is minimal. Suppose S is not minimal. Then there exists a vertex $u \in S$ such that $S - \{u\}$ is CCN-dominating set. Thus, u is CCN-adjacent to at least one vertex in $S - \{u\}$. Hence condition (i) dose not holed, also if $S - \{u\}$ is CCN-dominating set then every vertex $v \in V - S$ is CCN-adjacent to at least one vertex in $S - \{u\}$, That means condition (ii) does not hold. So we get contradiction. Hence S is minimal CCN-dominating set. ☐

Definition 2.7. The CCN-boundary of a set S is the set $B_{ccn}(S) = \{v \in V : |N_{ccn}[v] \cap S| = 1\}$, that is, the set of vertices CCN-dominated by exactly one vertex in S.

Theorem 2.3. *The CCN-dominating set S is a minimal CCN-dominating set if and only if $B_{ccn}(S)$ CCN-dominates S.*

Proof. Let S be minimal CCN-dominating set, let $u \in S$. Then by Theorem (2.2), we have tow cases

Case 1 : u is not CCN–adjacent to any vertex in S, so $u \in B_{ccn}(S)$.
Case 2: There exists a vertex $v \in V - S$ such that $N_{ccn}(v) \cap S = \{u\}$, so $v \in B_{ccn}(S)$ and u CCN-adjacent to v. Hence $B_{ccn}(S)$ CCN-dominates S.

Conversely, Let $B_{ccn}(S)$ be CCN-dominates S. Suppose that S is no minimal CCN-dominating set, the there exists $u \in S$, such that $S - \{u\}$ is CCN-dominating set. Since $B_{ccn}(S)$ CCN-dominates S, then there exists $v \in B_{ccn}(S)$ such that v CCN-dominate u, so $N_{ccn}[v] \cap S = \{u\}$, thus $v \notin S - \{u\}$, since $S - \{u\}$ is CCN-dominating set, then there exists $w \in S - \{u\}$ such that w CCN-adjacent to v, that is $w \in N_{ccn}[v] \cap S = \{u\}$ which is contradiction with $w \neq u$. Hence S is a minimal CCN-dominating set. ☐

Theorem 2.4. *A graph G has a unique minimal co-common neighborhood dominating set if and only if the set of all co-common neighborhood isolated vertices forms a co-common neighborhood dominating set.*

Proof. Let G has a unique minimal co-common neighborhood dominating set S, and suppose $I_{ccn} = \{u \in V : u \;\; is \;\; CCN - isolated \;\; vertex\}$. Then $I_{ccn} \subseteq S$, now suppose $S - I_{ccn} \neq \phi$, let $v \in S - I_{ccn}$, since v is not co-common neighborhood isolated vertex, $V - \{v\}$ is co-common neighborhood dominating set. Hence there exists a minimal co-common neighborhood dominating set $S_1 \subseteq V - \{v\}$ and $S_1 \neq S$ a contradiction to the fact that G has a unique minimal co-common neighborhood dominating set.

Conversely, if the set of all co-common neighborhood isolated vertices forms a co-common neighborhood dominating set, then it is clear that G has a unique minimal co-common neighborhood dominating set. ☐

Theorem 2.5. *Let G be a graph without co-common neighborhood isolated vertices. If S is minimal co-common neighborhood dominating set, then $V - S$ is CCN-dominating set.*

Proof. Let S be a minimal CCN-dominating set of G. Suppose $V-S$ is not CCN-dominating set. Then there exists a vertex u in S such that u is not CCN-adjacent to any vertex in $V-S$. But u is not CCN-isolated ,then u is CCN-adjacent to at least one vertex in $S-\{u\}$. Thus, $S-\{u\}$ is a co-common neighborhood dominating set of G, which contradicts the minimal co-common neighborhood dominating of S. Thus, every vertex in S is CCN-adjacent with at least one vertex in $V-S$. Hence $V-S$ is CCN-dominating set. □

Corollary 2.6. *Let G be a strongly regular graph with the parameters (n,k,λ,μ) such that $n>2k-\lambda$. If S is a minimal co-common neighborhood dominating set. Then $V-S$ is a dominating set.*

Theorem 2.7. *If G is a graph of order $n\geq 2$, then $2\leq \gamma_{ccn}(G)\leq n$.*

Proof. Since $\gamma_{ccn}(K_n)=n$ then $\gamma_{ccn}(G)\leq n$.

Now, let $\gamma_{ccn}(G)=1$ and S is $\gamma_{ccn}-set$, say $S=\{v\}$. Let $u\in V-S$, then $uv\in E(G)$ and $\left|\Gamma'(u,v)\right|\geq 1$, so there is $w\in V-\{u,v\}$ which is no adjacent to both u and v, that is w no CCN-adjacent to any element in S, which is contradiction, so $2\leq\gamma_{ccn}(G)$. □

Theorem 2.8. *For any graph G with n vertices, $\frac{n}{1+\Delta_{ccn}}\leq\gamma_{ccn}(G)$.*

Proof. Let S be a $\gamma_{ccn}-set$. Then for each $u\in S$, we have $|N_{ccn}(u)|\leq\Delta_{ccn}(G)$. Thus $|N_{ccn}(S)|\leq\gamma_{ccn}(G)\Delta_{ccn}(G)$. So

$$\begin{aligned} n &= |N_{ccn}[S]| \\ &= |S\cup N_{ccn}(S)| \\ &\leq \gamma_{ccn}(G)+\gamma_{ccn}(G)\Delta_{ccn}(G) \\ &= \gamma_{ccn}(G)(1+\Delta_{ccn}(G)) \end{aligned}$$

Hence

$$\frac{n}{1+\Delta_{ccn}(G)}\leq\gamma_{ccn}(G).$$

□

Corollary 2.9. *For any strongly regular graph with the parameters (n,k,λ,μ) we have $\frac{n}{1+k}\leq\gamma_{ccn}(G)$.*

Proposition 2.2. *If a graph G has no CCN-isolated vertices, then $\gamma_{ccn}(G)\leq\frac{n}{2}$.*

Proposition 2.3. *If G is a disconnected graph with components $G_1,G_2,...,G_r$, then $\gamma_{ccn}(G)=\gamma(G)$.*

Lemma 2.10. *For any graph G we have $\gamma_{ccn}(G)\geq\gamma(G)$.*

Lemma 2.11. *Let G be a strongly regular graph with parameters (n,k,λ,μ). Then for any vertices $u,v\in V$ we have*

$$\left|\Gamma'(u,v)\right|=\begin{cases} n-2k+\lambda & ,if\quad uv\in E(G); \\ n-2k+\mu-2 & ,if\quad uv\notin E(G). \end{cases}$$

Theorem 2.12. *Let G be a strongly regular graph with parameters (n,k,λ,μ) such that $n>2k-\lambda$. Then $\gamma_{ccn}(G)=\gamma(G)$.*

Proof. Let S be $\gamma - set$ and $v \in V - S$, then there exists $u \in S$ such that $uv \in E(G)$. So by Lemma (2.10), we have

$\left|\Gamma'(u,v)\right| = n-2k+\lambda$, since $n > 2k-\lambda$, then $\left|\Gamma'(u,v)\right| \geq 1$, so S is CCN-dominating set. Therefore $\gamma_{ccn}(G) \leq \gamma(G)$. So by Lemma (2.10), we have $\gamma_{ccn}(G) = \gamma(G)$. □

Theorem 2.13. *Let G be a graph with $\gamma(G) \geq 3$. Then $\gamma_{ccn}(G) = \gamma(G)$.*

Proof. Let S be $\gamma - set$ and $v \in V - S$. Then there exists $u \in S$ such that $uv \in E(G)$. If $\left|\Gamma'(u,v)\right| = 0$, then $\{u,v\}$ is dominating set which contradiction with $\gamma(G) \geq 3$. So $\left|\Gamma'(u,v)\right| \geq 1$, therefore S is CCN-dominating set, hance $\gamma_{ccn}(G) \leq \gamma(G)$. By Lemma (2.10), we have $\gamma_{ccn}(G) = \gamma(G)$. □

Theorem 2.14. *Let G be a graph with $\gamma(G) = 1$ and let $\{u_1\}, \{u_2\}, ...\{u_m\}$ be all $\gamma - set$ in G. Then*

$$\gamma_{ccn}(G) = \gamma_{ccn}(G - \{u_1, u_2, ..., u_m\}) + m.$$

Proof. Since $u_1, u_2, ..., u_m$ are CCN-isolated vertices and any CCN-dominating set contain all CCN-isolated vertices then $\gamma_{ccn}(G) = \gamma_{ccn}(G - \{u_1, u_2, ..., u_m\}) + m$. □

Theorem 2.15. *Let G be a graph with $\gamma(G) = 1$. Then $\gamma_{ccn}(G) \geq 3$.*

Proof. Let $\{u\}$ be a dominating set in G. Then

$$\begin{aligned} \gamma_{ccn}(G) &= \gamma_{ccn}(G-u)+1 \\ &\geq 2+1=3 \qquad by\, Theorem(2.7) \end{aligned} \tag{2.2}$$

□

Proposition 2.4. *Let G be a graph with $\gamma(G) \leq 2$ and let $e_1, e_2, ...e_m$ be all edges with endpoints represent a dominating set, then*

$$\gamma_{ccn}(G) = \gamma(G - \{e_1, e_2, ..., e_m\}.$$

Proposition 2.5. *Let G be a graph with $\gamma(G) = 2$ and for every $e = uv$ edge in G, $\{u,v\}$ is not dominating set. Then $\gamma_{ccn}(G) = 2$.*

Theorem 2.16. *Let G be a graph with $\gamma_{ccn}(G) = 2$. Then $\gamma(G) = 2$.*

Proof. Let $\gamma_{ccn}(G) = 2$. If $\gamma(G) \neq 2$, then $\gamma(G) = 1$ or $\gamma(G) \geq 3$, so by Theorems $(2.15, 2.13)$, we have $\gamma_{ccn}(G) \geq 3$ which is contradiction, so $\gamma(G) = 2$. □

The converse of Theorem (2.16) is not true because $\gamma(K_{3,4}) = 2$. but $\gamma_{ccn}(K_{3,4}) = 7$.

Proposition 2.6. *Let G be a graph of order n. Then $\gamma_{ccn}(G) = n$ if and only if the endpoints of any edges represent a dominating set in G.*

Corollary 2.17. $\gamma_{ccn}(K_n) = n$, $\gamma_{ccn}(C_4) = n$, $\gamma_{ccn}(S_n) = n$.

Theorem 2.18. *Let G be a bipartite graph with partition sets X, Y and for every vertices $u \in X$ and $v \in Y$ such that* $\deg(u) < |Y|$ *and* $\deg(v) < |X|$. *Then $\gamma_{ccn}(G) = \gamma(G)$.*

Proof. Let S be $\gamma - set$ and $v \in V - S$, then there exists $u \in S$ such that $uv \in E(G)$. then we have two cases

Case 1: $v \in X$, so $u \in Y$, since $\deg(v) < |Y|$, then there exists $w \in Y$ not adjacent to both u and v. Hence u CCN-adjacent to v.

Case 2: $v \in Y$, so $u \in X$, since $\deg(v) < |X|$, then there exists $w \in X$ not adjacent to both u and v. Hence u CCN-adjacent to v. Therefore S is CCN-dominating set, so $\gamma_{ccn}(G) \leq \gamma(G)$. By Lemma (2.10), we have $\gamma_{ccn}(G) = \gamma(G)$. □

Theorem 2.19. *Let G be a graph contains at least one isolated vertex. Then $\gamma_{ccn}(G) = \gamma(G)$.*

Proof. Let $x \in V$ be isolated vertex in G. Let S be γ -set and $v \in V - S$. Then there exists $u \in S$ such that $uv \in E(G)$. Since x is not adjacent to both u, v then $\left|\Gamma'(v, u)\right| \geq 1$, so S is CCN-dominating set, that is $\gamma_{ccn}(G) \leq \gamma(G)$. By lemma (2.10) we have $\gamma_{ccn}(G) = \gamma(G)$.. □

Theorem 2.20. *Let G be a connected graph with $red(G) \geq 3$. Then $\gamma_{ccn}(G) = \gamma(G)$.*

Proof. Let S be $\gamma - set$ and $v \in V - S$. Then there exists $u \in S$ such that $uv \in E(G)$. Since $ac(v) \geq 3$, then there exists $x \in V$ such that $d(x, v) \geq 3$, so x not adjacent to both v and u, that is $\left|\Gamma'(v, u)\right| \geq 1$, thus u CCN-adjacent to v. Therefore S is CCN-dominating set , hance $\gamma_{ccn}(G) \leq \gamma(G)$, by Lemma (2.20) we have $\gamma_{ccn}(G) = \gamma(G)$. □

Theorem 2.21. *Let $G_1 = (V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. Then*

$$\gamma_{ccn}(G_1 + G_2) = \gamma_{ccn}(G_1) + \gamma_{ccn}(G_2).$$

Proof. Let S_1, S_2 be CCN-dominating sets of G_1, G_2 respectively with $|S_1| = \gamma_{ccn}(G_1), |S_2| = \gamma_{ccn}(G_2)$. Then $S_1 \cup S_2$ is CCN-dominating set of $G_1 + G_2$, so

$$\gamma_{ccn}(G_1 + G_2) \leq |S_1 \cup S_2| = |S_1| + |S_2| = \gamma_{ccn}(G_1) + \gamma_{ccn}(G_2)$$

Similarly, let D be $\gamma_{ccn} - set$ in $G_1 + G_2$ and let $S_i = D \cap V_i$, $(i = 1, 2)$. Since every vertices in V_1 not CCN-adjacent to any vertices in V_2. Then S_1, S_2 are CCN-dominating sets of G_1, G_2 respectively, so

$$\gamma_{ccn}(G_1) + \gamma_{ccn}(G_2) \leq |S_1| + |S_2| = |D| = \gamma_{ccn}(G_1 + G_2)$$

Hence $\gamma_{ccn}(G_1 + G_2) = \gamma_{ccn}(G_1) + \gamma_{ccn}(G_2)$. □

Proposition 2.7. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with n_1, n_2 vertices respectively such that $n_1 \geq 2$. Then*

$$\gamma_{ccn}(G_1 \circ G_2) = \gamma(G_1 \circ G_2) = n_1.$$

Definition 2.8. Let G be a graph with $\Delta(G) \neq \delta(G)$. The maximum degree of vertices less than $\Delta(G)$ is called maximal degree and denoted by $\sigma(G)$. In case $\Delta(G) = \delta(G)$ we define $\sigma(G)$ as $\sigma(G) = \Delta(G) = \delta(G)$.

Example 2.22. $\sigma(P_n) = 1$, $\sigma(C_n) = 2$, $\sigma(W_n) = 3$.

Theorem 2.23. *Let G be a graph of order $n \geq 3$ with $\delta(G) = 1$, $\Delta(G) = n - 1$. Then*

$$3 \leq \gamma_{ccn}(G) \leq n - \sigma(G) + 1.$$

Proof. Let G be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$. Without loss generally let $\deg(v_1) = n-1$, $\deg(v_2) = 1$. Clearly all edges such that it endpoints represent a dominating set are $e_2, e_3, ..., e_n$ where $e_i = v_1 v_i$, $2 \leq i \leq n$, so by Proposition (2.25) $\gamma_{ccn}(G) = \gamma(G - \{e_2, e_3, ..., e_n\})$. Since $\Delta(G - \{e_2, e_3, ..., e_n\}) = \sigma(G) - 1$, and $\gamma(G - \{e_2, e_3, ..., e_n\}) \leq n - \Delta(G - \{e_2, e_3, ..., e_n\})$, then $\gamma_{ccn}(G) \leq n - \sigma(G) + 1$. Since $\gamma(G) = 1$, then by Theorem (2.15) $3 \leq \gamma_{ccn}(G)$. Hence $3 \leq \gamma_{ccn}(G) \leq n - \sigma(G) + 1$. □

Remark 2.1.

i. If $\sigma(G) = 1$, $\Delta(G) = n-1$ and $\delta(G) = 1$, then $\gamma_{ccn}(G) = n$.

ii. If $\sigma(G) = n-2$, $\Delta(G) = n-1$ and $\delta(G) = 1$, then $\gamma_{ccn}(G) = 3$.

iii. If $\sigma(G) = n-3$, $\Delta(G) = n-1$ and $\delta(G) = 1$, then $\gamma_{ccn}(G) = 4$.

Definition 2.9. A subset S of the vertex set in a graph G is said to be CCN-independent set if no tow vertices in S are CCN-adjacent in G, *i.e* for every vertex $u, v \in S$, $uv \notin E(G)$ or $\left|\Gamma'(u,v)\right| = 0$. The maximum cardinality of CCN-independent set is denoted by β_{ccn}.

Definition 2.10. A CCN-independent set S is called maximal if any vertex set properly containing S is not CCN-independent set. The lower CCN-independence number i_{ccn} is the minimum cardinality of the maximal CCN-independent set.

Example 2.24. *In figure 1, the set $\{1, 2, 4\}$ is CCN-independent and $\beta_{ccn} = 3, i_{ccn} = 3$.*

Theorem 2.25. *Let S be a maximal CCN-independent set. Then S is minimal CCN-dominating set.*

Let S be a maximal CCN -independent set and let $v \in V - S$. If v is no CCN-adjacent to any vertex in S, then $S \cup \{v\}$ is CCN-independent set which is a contradiction because S is maximal. So there exists vertex $u \in S$ such that $uv \in E(G)$ and $\left|\Gamma'(u,v)\right| \geq 1$, hence S is CCN-dominating set. Now, if there is $v \in S$, such that $S - \{v\}$ is CCN-dominating set, then there exists $u \in S$ such that $uv \in E(G)$ and $\left|\Gamma'(u,v)\right| \geq 1$, a contradiction since S is CCN-independent set. So S is minimal CCN-dominating set.

Proposition 2.8. *For any graph G , $\gamma_{ccn} \leq i_{ccn} \leq \beta_{ccn} \leq \Gamma_{ccn}$.*

3 Total CCN-Dominating Set and CCN-Covering Set

Definition 3.1. A subset $S \subseteq V$ is said to be total co-common neighborhood dominating set (total CCN-dominating set) if for all $v \in V$, there exist $u \in S$ such that $uv \in E(G)$ and $\left|\Gamma'(u,v)\right| \geq 1$.

If G has no CCN-isolated points, then V is total CCN-dominating set. The minimum cardinality of a total CCN-dominating set in a graph G is called the total CCN-domination number of G denoted by γ_{tccn} . If G has a CCN-isolated point then we take $\gamma_{tccn} = \infty$.

Proposition 3.1.

(i)

$$\gamma_{tccn}(K_n) = \infty.$$

(ii)

$$\gamma_{tccn}(W_n) = \infty.$$

(iii)

$$\gamma_{tccn}(K_{a,b}) = \infty.$$

(iv)

$$\gamma_{tccn}(P_n, n \geq 5) = \begin{cases} \frac{n}{2} & , if \quad n \equiv 0(mod4); \\ \lfloor \frac{n}{2} \rfloor + 1 & , otherwise. \end{cases}$$

(v)

$$\gamma_{tccn}(C_n, n \geq 5) = \begin{cases} \frac{n}{2} & , if \quad n \equiv 0(mod4); \\ \lfloor \frac{n}{2} \rfloor + 1 & , otherwise. \end{cases}$$

Definition 3.2. An edge $e = uv \in E(G)$ is said to be co-common neighborhood edge (CCN-edge) if $\left| \Gamma'(u, v) \right| \geq 1$.

Definition 3.3. Let $G = (V, E)$ be a graph. A subset S of V is called co-common neighborhood vertex covering (CCN-vertex covering) of G if for each CCN-edge $e = uv$, either $u \in S$ or $v \in S$. The minimum cardinality of CCN-vertex covering of G is called the CCN-covering number of G and denoted by $\alpha_{ccn}(G)$. If G has no CCN-edge then $\alpha_{ccn}(G) = 0$.

Proposition 3.2.

(i)

$$\alpha_{ccn}(K_n) = 0.$$

(ii)

$$\alpha_{ccn}(P_n) = \left\lceil \frac{n-1}{2} \right\rceil, \quad (n \geq 4).$$

(iii)

$$\alpha_{ccn}(C_n) = \left\lceil \frac{n}{2} \right\rceil, \quad (n \geq 5)\ .$$

(iv)

$$\alpha_{ccn}(K_{n,m}) = 0.$$

(v)

$$\alpha_{ccn}(W_n) = \left\lceil \frac{n-1}{2} \right\rceil, \quad (n \geq 6).$$

Theorem 3.1. *Let G be any graph has no CCN-isolated vertex. Then every CCN-vertex cover is CCN-dominating set of G, i.e. $\gamma_{ccn}(G) \leq \alpha_{ccn}(G)$.*

Proof. Let S be CCN-vertex cover and let $v \in V - S$. Since v is no CCN-isolated vertex, then there exist $u \in V$ such that $e = uv$ is CCN-edge, since S is CCN-vertex cover and $v \notin S$, then $u \in S$, so S is CCN-dominating set. □

Theorem 3.2. *Let G be any graph. Then S is CCN-vertex cover if and only if $V - S$ is CCN-independent set.*

Proof. Let S be CCN-vertex cover and $u, v \in V - S$. If uv is CCN-edge, then $u \in S$ or $v \in S$ which is contradiction because $u, v \notin S$, so $uv \notin E(G)$ or $\left| \Gamma'(u, v) \right| = 0$ that is, $v - S$ is CCN-independent set.

Conversely, let $V - S$ be CCN-independent set and uv be CCN-edge. Then $u \notin V - S$ or $v \notin V - S$, that is $u \in S$ or $v \in S$, so S is CCN-vertex cover.

□

Corollary 3.3. *If S is CCN-independent set then $v - S$ is CCN-vertex cover.*

Theorem 3.4. *Let G be a graph of order n. Then*

$$\alpha_{ccn}(G) + \beta_{ccn}(G) = n$$

Proof. Let S be $\alpha_{ccn} - set$. Then by Theorem (3.2), $V - S$ is CCN-independent set, so $|V - S| \leq \beta_{ccn}(G)$, thus

$$n \leq \alpha_{ccn}(G) + \beta_{ccn}(G) \qquad (1)$$

Let D be $\beta_{ccn} - set$, then by corollary (3.3) $V - D$ is CCN-vertex cover, so $\alpha_{ccn}(G) \leq |V - D|$, thus

$$\alpha_{ccn}(G) + \beta_{ccn}(G) \leq n \qquad (2)$$

From 1 and 2 we have

$$\alpha_{ccn}(G) + \beta_{ccn}(G) = n.$$

□

4 CCN-Domatic Number

Definition 4.1. Let G be a graph without CCN-isolated vertices. A co-common neighborhood domatic partition (CCN-domatic partition) of G is a partition $\{V_1, V_2, ..., V_k\}$ of $V(G)$ in which each V_i is CCN-dominating set of G . The CCN-domatic number is the maximum order of an CCN-domatic partition of G and is denoted by $d_{ccn}(G)$.

Example 4.1. *The CCN-domatic number of the graph G (peterson graph) in figure(**??**) is $d_{ccn}(G) = 2$, because $\{\{1, 3, 4, 5, 7\}, \{2, 6, 8, 9, 10\}\}$ is CCN-domatic partition of G of maximum order.*

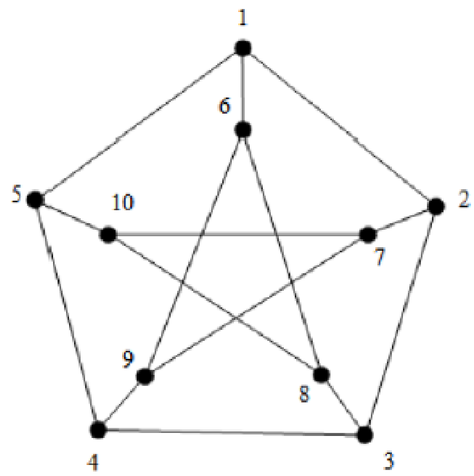

Fig. 2. Petersen graph

Definition 4.2. A graph G is called CCN-domatically full if $d_{ccn}(G) = \delta_{ccn}(G) + 1$.

Example 4.2.

1. *C_6 is CCN-domatically full because $d_{ccn}(G) = 3$ and $\delta_{ccn}(G) = 2$..*
2. *C_5 is not CCN-domatically full because $d_{ccn}(G) = 2$ and $\delta_{ccn}(G) = 2$.*

5 Conclusion

In this research work, we have introduced a new type of domination of graphs called the co-common neighborhood domination, and we have studied some basic properties and relations between this type of domination and some other domination parameters. Still there are many problems and related work for this new type of domination for the future research like connected co-common neighborhood domination, independent co-common neighborhood domination, total co-common neighborhood domination, and inverse co-common neighborhood domination of graphs.

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Competing Interests

Authors have declared that no competing interests exist.

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