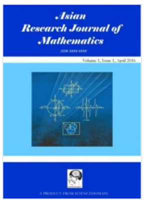

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The New Iterative Method for Approximate Solutions of Time Fractional Kdv, K(2,2), Burgers, and Cubic Boussinesq Equations

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Authors' contributions

Both authors contributed equally and significantly to writing this paper. Both authors read and approved the final manuscript.

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Abstract

In this paper, new iterative method is used to determine approximate solutions for the time fractional KdV, the K(2,2), the Burgers, and the cubic Boussinesq equations. The obtained approximate solutions are compared with the exact results. The study reveals that the present method is very effective, accurate and convenient.

Keywords: New iterative method; time-fractional KdV equation; K(2,2) equation; Burgers equation; cubic Boussinesq equation; approximate solution.

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1 Introduction

Over the past thirty years, researchers have paid attention towards fractional differential equations (FDEs), because many physical phenomena in science and engineering can be modeled using the fractional derivatives. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which combines and generalizes the concept of integer-order differentiation and n-fold integration. Various branches of sciences and engineering benefit of these equations such as fluid mechanics, entropy and engineering, viscoelastic materials, physics, chemistry and signal processing [1, 2, 3, 4, 5, 6].

Nonlinear phenomena play a crucial role in applied mathematics and physics. Specially the theory of solitons which is studied in various forms such as analysing topological solitons known as shock-waves, singular solitons that are also known as rogue waves in oceanography and optical rogons in nonlinear optics. KdV equation is the pioneering equation which gives solitary wave solutions [7, 8, 9, 10, 11]. The K(n, n) equation proposed in [12] is the original equation for compactions. The Burgers equation appears in fluid mechanics. This equation incorporates both convection and diffusion in fluid dynamics, and is used to describe the structure of shock waves. However, the cubic Boussinesq gives rise to solitons and appeared in the works of Priestly and Clarkson [13].

Many nonlinear partial fractional differential equations can be solved by various methods such as Adomian decomposition method ADM[14, 15], Variational iteration method VIM [16], Homotopy-perturbation method HPM [17], Homotopy analysis method [18], Finite element method [19]. In this work, the new iterative method (NIM) which is one of the most reliable and effective technique suggested by Daftardar-Gejji and Jafari [20, 21, 22] is used to obtain an approximate solutions of nonlinear dispersive time fractional KdV, the K(2,2), the Burgers, and the cubic Boussinesq equations. The substantial amount of research work has been carried out on these nonlinear dispersive equations [23, 24, 25, 26, 27].

The rest of this paper is organized as follows. In Section 2, basic definitions are presented. In Section 3 we give an analysis of the new iterative method. The numerical results and graphs for the time fractional KdV equation, K(2,2) equation, Burgers equation and Cubic Boussinesq equation are presented in Section 4. Finally, we give our conclusions in Section 5.

2 Basic Definitions

Definition 2.1.[1] The left sided Riemann-Liouville fractional integral of order α, $\alpha \geq 0$ of function $f \in C_\mu$ and $\mu \geq -1$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau; \qquad t > 0, \alpha > 0 \tag{2.1}$$

Definition 2.2.[1] The caputo fractional derivative of f , $f \in C_{-1}^m$, $m \in N \cup \{0\}$ is defined as

$$\begin{aligned} D_t^\alpha f(x,t) &= \frac{\partial^m f(x,t)}{\partial t^m}, \qquad \alpha = m \\ &= I_t^{m-\alpha} \frac{\partial^m f(x,t)}{\partial t^m}, \qquad m-1 < \alpha \leq m, \quad m \in N \end{aligned} \tag{2.2}$$

Note that

$$I_t^\alpha D_t^\alpha f(x,t) = f(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k f(x,0)}{\partial t^k} \frac{t^k}{k!} \qquad m-1 < \alpha \leq m, m \in N \tag{2.3}$$

$$I_t^\alpha t^\beta = \frac{t^{\alpha+\beta}\ \Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \tag{2.4}$$

3 The New Iterative Method

Daftardar-Gejji and Jafari [20] have proposed a simple technique to solve nonlinear functional equations known as a new iterative method (NIM) in which a couple of computer commands are sufficient to calculate approximate solution.
Consider a functional equation of the form

$$u(\bar{x},t) = f(\bar{x},t) + L(u(\bar{x},t)) + N(u(\bar{x},t)) \tag{3.1}$$

Where f is a given function, L and N are given linear and non-linear operator of u respectively, $\bar{x} = (x_1, x_2, x_3, \ldots, x_n)$ Let u be a solution of eqn (3.1) having the series form:

$$u(\bar{x},t) = \sum_{i=0}^{\infty} u_i(\bar{x},t) \tag{3.2}$$

Since L is linear

$$L(\sum_{i=0}^{\infty} u_i) = \sum_{i=0}^{\infty} L(u_i)$$

The nonlinear operator here is decomposed as :

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \tag{3.3}$$

$$= \sum_{i=0}^{\infty} G_i \tag{3.4}$$

where $G_0 = N(u_0)$ and $G_i = \left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}$, $i \geq 1$

Hence eqn (3.1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} G_i \tag{3.5}$$

Further define the recurrence relation :

$$u_0 = f$$

$$u_1 = L(u_0) + G_0$$

$$u_{m+1} = L(u_m) + G_m, \quad m = 1, 2, \cdots \tag{3.6}$$

Then

$$(u_1 + u_2 + \cdots + u_{m+1}) = L(u_0 + u_1 + \cdots + u_m) + N(u_0 + u_1 + \cdots + u_m), \qquad m = 1, 2, \cdots$$

and $u = f + \sum_{i=1}^{\infty} u_i$

The condition for convergence of the series $\sum u_i$ is presented in [28].

4 Numerical Application

In this section, we test the efficiency of the NIM by applying it on some nonlinear fractional differential equations. All computations are performed using Mathematica .

Example 1. We consider first time fractional KdV equation in the form.

$$u_t^\alpha - 3(u^2)_x + u_{xxx} = 0; \qquad 0 < \alpha \le 1 \tag{4.1}$$

with initial condition $u(x,0) = 6x$

The exact solution to the above classical initial value problem is given by [23]

$$u(x,t) = \frac{6x}{1-36t}, \qquad |36t| < 1$$

Applying Integral operator I^α both side of Eq.(4.1) and using initial condition we obtain the relation

$$u(x,t) = x + L(u) + N(u)$$

where $L(u) = I^\alpha\{-u_{xxx}\}$ and $N(u) = I^\alpha\{3(u^2)_x\}$

Taking series solution as $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ and using (3.3) and (3.6)

$u_0 = 6x$ Applying NIM successively we get

$$u_1 = \frac{216\ x\ t^\alpha}{\Gamma(1+\alpha)} \tag{4.2}$$

$$u_2 = \frac{279936\ x\ \Gamma(1+2\alpha) t^{3\alpha}}{\Gamma(1+\alpha)^2\ \Gamma(1+3\alpha)} + \frac{15552\ x\ t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{4.3}$$

$$\begin{aligned} u_3 = & \frac{1451188224\ x\ \Gamma(1+4\alpha) t^{5\alpha}}{\Gamma(1+2\alpha)^2\ \Gamma(1+5\alpha)} + \frac{40310784\ x\ \Gamma(1+3\alpha) t^{4\alpha}}{\Gamma(1+\alpha)\ \Gamma(1+2\alpha)\ \Gamma(1+4\alpha)} + \frac{1119744\ x\ t^{3\alpha}}{\Gamma(1+3\alpha)} \\ & + \frac{470184984576\ x\ \Gamma(1+6\alpha)\ \Gamma(1+2\alpha)^2\ t^{7\alpha}}{\Gamma(1+\alpha)^4\ \Gamma(1+3\alpha)^2\ \Gamma(1+7\alpha)} + \frac{52242776064\ x\ \Gamma(1+5\alpha) t^{6\alpha}}{\Gamma(1+\alpha)^2\ \Gamma(1+3\alpha)\ \Gamma(1+6\alpha)} \\ & + \frac{725594112\ x\ \Gamma(1+4\alpha)\ \Gamma(1+2\alpha)\ t^{5\alpha}}{\Gamma(1+\alpha)^3\ \Gamma(1+3\alpha)\ \Gamma(1+5\alpha)} + \frac{20155392\ x\ \Gamma(1+2\alpha)\ t^{4\alpha}}{\Gamma(1+\alpha)^2\ \Gamma(1+4\alpha)} \end{aligned} \tag{4.4}$$

In the same manner the remaining components of the iteration formula (3.6) can be obtained from Mathematica software. we get five term approximate solution of Eq. (4.1) as

$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t)$

Example 2. We next consider time fractional K(2,2) equation in the form.

$$u_t^\alpha + (u^2)_x + (u^2)_{xxx} = 0; \qquad 0 < \alpha \le 1 \tag{4.5}$$

with initial condition, $u(x,0) = x$

The exact solution to the above classical initial value problem is given by [23]

$$u(x,t) = \frac{x}{1+2t},$$

Applying Integral operator I^α both side of Eq. (4.5) and using initial condition we obtain the relation

$$u(x,t) = x + L(u) + N(u)$$

where $N(u) = I^{\alpha}\{-(u^2)_x - (u^2)_{xxx}\}$

Taking series solution as $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ and using (3.3) and (3.6)

$u_0 = x$ Applying NIM successively we get

$$u_1 = \frac{-2\ x\ t^{\alpha}}{\Gamma(1+\alpha)} \tag{4.6}$$

$$u_2 = \frac{-8\ x\ \Gamma(1+2\alpha)t^{3\alpha}}{\Gamma(1+\alpha)^2\ \Gamma(1+3\alpha)} + \frac{8\ x\ t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{4.7}$$

$$\begin{aligned} u_3 = & \frac{-128\ x\ \Gamma(1+4\alpha)t^{5\alpha}}{\Gamma(1+2\alpha)^2\ \Gamma(1+5\alpha)} + \frac{64\ x\ \Gamma(1+3\alpha)t^{4\alpha}}{\Gamma(1+\alpha)\ \Gamma(1+2\alpha)\ \Gamma(1+4\alpha)} - \frac{32\ x\ t^{3\alpha}}{\Gamma(1+3\alpha)} \\ & - \frac{128\ x\ \Gamma(1+6\alpha)\ \Gamma(1+2\alpha)^2\ t^{7\alpha}}{\Gamma(1+\alpha)^4\ \Gamma(1+3\alpha)^2\ \Gamma(1+7\alpha)} + \frac{256\ x\ \Gamma(1+5\alpha)t^{6\alpha}}{\Gamma(1+\alpha)^2\ \Gamma(1+3\alpha)\ \Gamma(1+6\alpha)} \\ & - \frac{64\ x\ \Gamma(1+4\alpha)\ \Gamma(1+2\alpha)\ t^{5\alpha}}{\Gamma(1+\alpha)^3\ \Gamma(1+3\alpha)\ \Gamma(1+5\alpha)} + \frac{32\ x\ \Gamma(1+2\alpha)\ t^{4\alpha}}{\Gamma(1+\alpha)^2\ \Gamma(1+4\alpha)} \end{aligned} \tag{4.8}$$

In the same manner the remaining components of the iteration formula (3.6) can be obtained from Mathematica software. The five term approximate solution of Eq. (4.5) is

$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t)$

Example 3. We next consider time fractional Burgers equation as:

$$u_t^{\alpha} + \frac{1}{2}(u^2)_x - (u)_{xxx} = 0; \qquad 0 < \alpha \leq 1 \tag{4.9}$$

with initial conditions, $u(x,0) = x$

The exact solution to the above classical initial value problem is given by [23]

$$u(x,t) = \frac{x}{1+t},$$

Applying Integral operator I^{α} both side of Eq.(4.9) and using initial condition we obtain the relation

$$u(x,t) = x + L(u) + N(u)$$

where $L(u) = I^{\alpha}\{u_{xxx}\}$ and $N(u) = I^{\alpha}\{-\frac{1}{2}(u^2)_x\}$

Taking series solution as $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ and using (3.3) and (3.6) we get $u_0 = x$

Applying NIM successively we get

$$u_1 = \frac{-\ x\ t^{\alpha}}{\Gamma(1+\alpha)} \tag{4.10}$$

$$u_2 = \frac{-\ x\ \Gamma(1+2\alpha)t^{3\alpha}}{\Gamma(1+\alpha)^2\ \Gamma(1+3\alpha)} + \frac{2\ x\ t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{4.11}$$

$$u_3 = \frac{-4\ x\ \Gamma(1+4\alpha)t^{5\alpha}}{\Gamma(1+2\alpha)^2\ \Gamma(1+5\alpha)} + \frac{4\ x\ \Gamma(1+3\alpha)t^{4\alpha}}{\Gamma(1+\alpha)\ \Gamma(1+2\alpha)\ \Gamma(1+4\alpha)} - \frac{4\ x\ t^{3\alpha}}{\Gamma(1+3\alpha)}$$

$$- \frac{x\ \Gamma(1+6\alpha)\ \Gamma(1+2\alpha)^2\ t^{7\alpha}}{\Gamma(1+\alpha)^4\ \Gamma(1+3\alpha)^2\ \Gamma(1+7\alpha)} + \frac{4\ x\ \Gamma(1+5\alpha)t^{6\alpha}}{\Gamma(1+\alpha)^2\ \Gamma(1+3\alpha)\ \Gamma(1+6\alpha)}$$
$$- \frac{2\ x\ \Gamma(1+4\alpha)\ \Gamma(1+2\alpha)\ t^{5\alpha}}{\Gamma(1+\alpha)^3\ \Gamma(1+3\alpha)\ \Gamma(1+5\alpha)} + \frac{2\ x\ \Gamma(1+2\alpha)\ t^{4\alpha}}{\Gamma(1+\alpha)^2\ \Gamma(1+4\alpha)} \tag{4.12}$$

In the same manner the remaining components of the iteration formula (3.6) can be obtained from Mathematica software. The five term approximate solution of Eq.(4.9) is given as

$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t)$

Example 4. We finally consider the time fractional cubic Boussinesq equation.

$$u_t^\alpha - u_{xx} + 2(u^3)_{xx} - u_{xxxx} = 0; \qquad 0 < \alpha \leq 2 \tag{4.13}$$

with initial conditions, $u(x,0) = \frac{1}{x}$, $u_t(x,0) = -\frac{1}{x^2}$

The exact solution to the above initial value problem is given by [23]

$$u(x,t) = \frac{1}{x+t}$$

Applying Integral operator I^α both side of Eq.(4.13) and using initial condition we obtain the relation

$$u(x,t) = \frac{1}{x} - \frac{t}{x^2} + L(u) + N(u)$$

where $L(u) = I^\alpha\{u_{xx} + u_{xxxx}\}$ and $N(u) = I^\alpha\{-2(u^3)_{xx}\}$

Taking series solution as $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ and using (3.3) and (3.6)

$$u_0 = \frac{1}{x} - \frac{t}{x^2}$$

Applying NIM successively we get

$$u_1 = \frac{2t^\alpha}{x^3\Gamma(1+\alpha)} - \frac{6t^{1+\alpha}}{x^4\Gamma(2+\alpha)} - \frac{360t^{2+\alpha}}{x^7\Gamma(3+\alpha)} + \frac{504t^{3+\alpha}}{x^8\Gamma(4+\alpha)} \tag{4.14}$$

$$u_2 = \frac{360t^{2\alpha}}{x^7\Gamma(1+2\alpha)} + \frac{24t^{2\alpha}}{x^5\Gamma(1+2\alpha)} - \frac{3528t^{1+2\alpha}}{x^8\Gamma(2+2\alpha)} - \frac{120t^{1+2\alpha}}{x^6\Gamma(2+2\alpha)} + \frac{1008t^{1+2\alpha}\Gamma(2+\alpha)}{x^8\Gamma(1+\alpha)\Gamma(2+2\alpha)}$$
$$- \frac{1620000t^{2+2\alpha}}{x^{11}\Gamma(3+2\alpha)} - \frac{20160t^{2+2\alpha}}{x^9\Gamma(3+2\alpha)} - \frac{672t^{2+2\alpha}\Gamma(3+\alpha)}{x^9\Gamma(1+\alpha)\Gamma(3+2\alpha)} - \frac{4032t^{2+2\alpha}\Gamma(3+\alpha)}{x^9\Gamma(2+\alpha)\Gamma(3+2\alpha)}$$
$$+ \frac{3659040t^{3+2\alpha}}{x^{12}\Gamma(4+2\alpha)} + \frac{36288t^{3+2\alpha}}{x^{10}\Gamma(4+2\alpha)} + \frac{2592t^{3+2\alpha}\Gamma(4+\alpha)}{x^{10}\Gamma(2+\alpha)\Gamma(4+2\alpha)} - \frac{475200t^{3+2\alpha}\Gamma(4+\alpha)}{x^{12}\Gamma(3+\alpha)\Gamma(4+2\alpha)}$$
$$+ \frac{285120t^{4+2\alpha}\Gamma(5+\alpha)}{x^{13}\Gamma(3+\alpha)\Gamma(5+2\alpha)} + \frac{798336t^{4+2\alpha}\Gamma(5+\alpha)}{x^{13}\Gamma(4+\alpha)\Gamma(5+2\alpha)} - \frac{471744t^{5+2\alpha}\Gamma(6+\alpha)}{x^{14}\Gamma(4+\alpha)\Gamma(6+2\alpha)} + \dots \tag{4.15}$$

In the same manner the remaining components of the iteration formula (3.6) can be obtained from Mathematica software. The three term approximate solution of Eq. (4.13) is given by

$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t)$

In order to illustrate the efficiency of the NIM, we compare the Absolute error between exact solution [23] and approximate solution obtained by NIM for $\alpha = 1$ of Ex. 1–3 and $\alpha = 2$ of Ex.4

From Table 1 it has been observed that approximate solutions for $\alpha = 1$ of examples $1-3$ and $\alpha = 2$ of example 4 are very close to exact solutions.

In Fig. 1,3 and 5 we study the surfaces of 5th−order approximate solution obtained by NIM for $\alpha = 1$, and In Fig. 2,4 and 6 we study the plots obtained from the 5th −order NIM approximation at $\alpha = 1$ and $x = 1$,of examples $1-3$. In Fig. 7, we study the surfaces of 3rd−order approximate solution obtained by NIM for $\alpha = 2$, and In Fig. 8 we study the plots obtained from the 3rd −order NIM approximation at $\alpha = 2$ and $x = 1$,of example 4.

Table 1. Absolute errors for difference between exact solution and approximate solution obtained by NIM for $\alpha = 1$ of examples $1-3$ and for $\alpha = 2$ of example 4

t	x	Example 1	Example 2	Example 3	Example 4
0.002	0.5	9.30800×10^{-7}	6.76126×10^{-14}	2.16493×10^{-15}	2.97164×10^{-10}
	1.0	1.86160×10^{-6}	1.35225×10^{-13}	4.32987×10^{-15}	1.48569×10^{-13}
	1.5	2.79240×10^{-6}	2.02727×10^{-13}	6.43929×10^{-15}	1.88738×10^{-15}
0.006	0.5	3.35997×10^{-4}	1.61007×10^{-11}	5.10703×10^{-13}	2.16420×10^{-7}
	1.0	6.71995×10^{-4}	3.22014×10^{-11}	1.02141×10^{-12}	1.08333×10^{-10}
	1.5	1.00799×10^{-3}	4.83020×10^{-11}	1.53211×10^{-12}	1.30385×10^{-12}
0.01	0.5	6.72852×10^{-3}	2.03003×10^{-10}	6.50236×10^{-12}	4.63388×10^{-6}
	1.0	1.34570×10^{-2}	4.06006×10^{-10}	1.30053×10^{-11}	2.32069×10^{-9}
	1.5	2.01856×10^{-2}	6.09009×10^{-10}	1.95082×10^{-11}	2.79359×10^{-11}

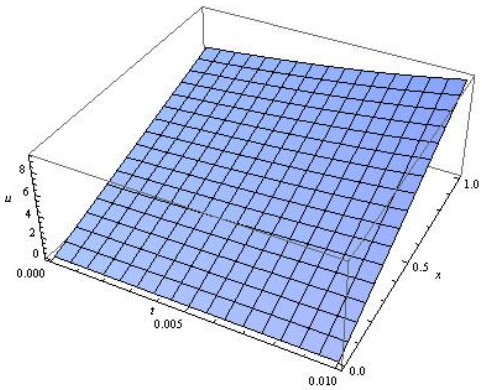

Fig. 1. Approx. soln of eqn (4.1),$\alpha = 1$

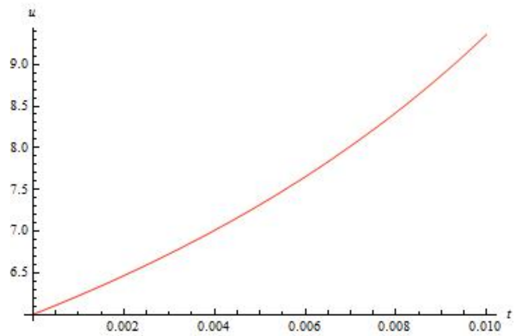

Fig. 2. Approx. soln $\alpha = 1, \ x = 1$

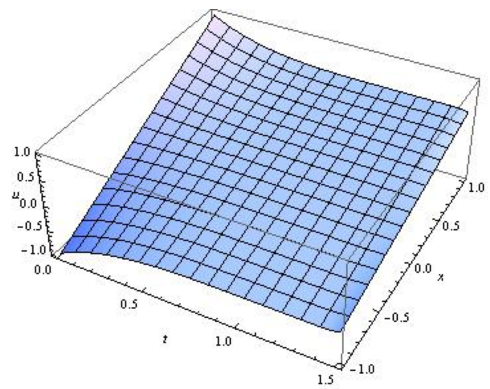

Fig. 3. Approx. soln of eqn (4.5), $\alpha = 1$

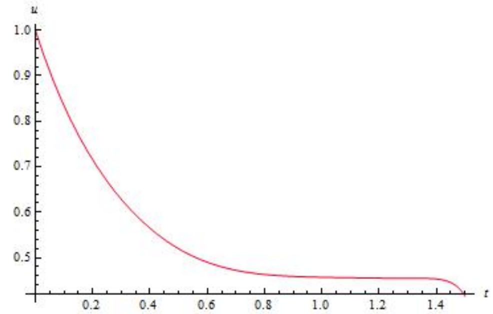

Fig. 4. Approx. soln $\alpha = 1,\ x = 1$

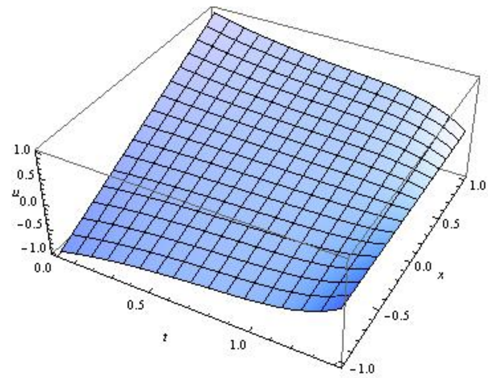

Fig. 5. Approx.soln of eqn (4.9),$\alpha = 1$

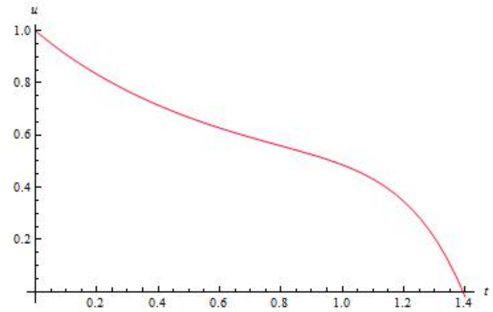

Fig. 6. Approx.soln $\alpha = 1,\ x = 1$

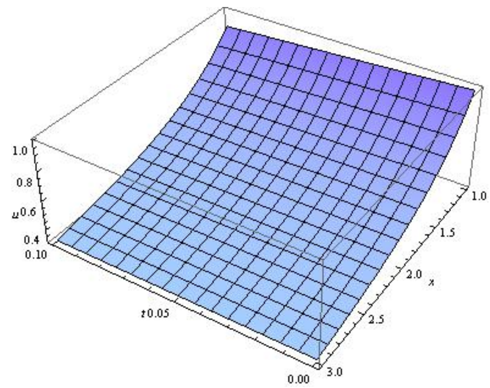

Fig. 7. Approx.soln of eqn (4.13),$\alpha = 2$

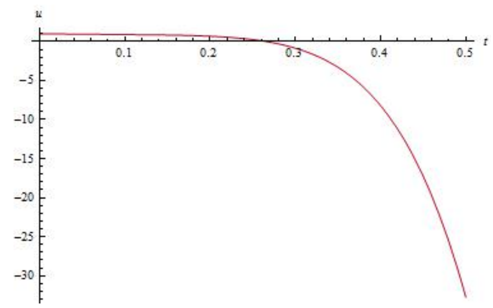

Fig. 8. Approx.soln $\alpha = 2,\ x = 1$

5 Conclusions

The numerical results showed that the new iterative method (NIM) is very reliable and efficient technique in finding approximate solutions as well as analytical solutions of many fractional physical models. To be precise, the approximate solutions obtained were in accordance with the exact solutions even if lower order approximations were used. The accuracy can be improved by using higher-order approximate solutions. The work emphasized our belief that the present method is a reliable technique to handle linear and nonlinear fractional differential equations.

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Competing Interests

Authors have declared that no competing interests exist.

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